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Surplus-Invariant, Law-Invariant, and Conic Acceptance Sets Must Be the Sets Induced by Value at Risk

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Abstract. The regulator is interested in proposing a capital adequacy test by specifying an acceptance set for firms' capital positions at the end of a given period. This set needs to be surplus invariant; i.e., not to depend on the surplus of firms' shareholders, because the test means to protect firms' liability holders. We prove that any surplus-invariant, law-invariant, and conic acceptance set must be the set of capital positions whose value at risk at a given level is less than zero. The result still holds if we replace conicity with numéraire invariance, a property stipulating that whether a firm passes the test should not depend on the currency used to denominate its assets.

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Keywords: capital adequacy tests • value at risk • surplus invariance • conicity • numéraire invariance

1. Introduction

Suppose a firm has initial capital c , debt d , and thus asset value $c + d$ at time 0. Suppose the firm invests its assets in a portfolio that generates net random return R in the period $[0, T]$. Suppose the firm needs to pay the interest rd together with the face value d to the creditor at time T , where r is the interest rate. Then, the profit and loss (P&L) of the firm at the end of the period is $Y = (c + d)R - rd$. The Basel II Accord (Basel Committee on Banking Supervision 2006, 2009) proposes a capital adequacy test based on the P&L: the value at risk (VaR) of the P&L at some confidence level, e.g., at 99%, is calculated and then the firm is required to hold as much capital as the amount of the VaR. Thus, if we denote $V@R_\alpha(Y)$ as the α -level VaR of Y , then the firm's position is acceptable if and only if $V@R_\alpha(Y) \leq c$, i.e., if and only if $V@R_\alpha(X) \leq 0$, where $X := Y + c = (c + d)(R + 1) - (1 + r)d$ is the capital position of the firm, i.e., the firm's assets net of its liability, at the end of the period.

As in the Basel Accords, the regulator is interested in proposing a capital adequacy test by specifying an acceptance set for firms' capital positions at the end of a given period. If a firm does not pass the test, it can (1) adjust its current portfolio, (2) inject new capital and hold it as cash, and (3) inject new capital and invest it in a portfolio. Then, the regulator can check if the updated capital position is acceptable by performing the capital adequacy test again.

The positive part of a firm's capital position is the surplus for the equity holders and the negative part is

defined as the *option to default* of the firm, which represents the portion of liabilities that is not paid off using the firm's assets. As argued by Staum (2013), Cont et al. (2013), Koch-Medina et al. (2015, 2017), the main purpose of a capital adequacy test is to protect liability holders; hence, whether a firm's capital position lies in the acceptance set should depend on only the negative part of the firm's capital position but not the positive part, a property referred to as surplus invariance.

In the present paper, we prove that surplus-invariant, law-invariant, and conic acceptance sets must be the sets induced by VaR, i.e., must be the set of capital positions whose VaR at a given level is less than zero. We assume neither convexity nor coherence, therefore the family of acceptance sets under our investigation include many that are not convex, such as those induced by the natural risk statistics proposed by Kou et al. (2013) and the distortion risk measures proposed by Kou and Peng (2016). Therefore our result is different from those in Koch-Medina et al. (2015, 2017).

Law-invariance is a commonly adopted property that allows the regulator to perform statistical tests on whether a firm uses a valid model in the capital adequacy test. Conicity, which means that scaling a capital position by any positive constant does not affect the acceptability of the position, is assumed in the notion of coherent risk measures (Artzner et al. 1999). Conicity is implied by numéraire invariance, a property that is introduced by Artzner et al. (2009) and further investigated by Koch-Medina et al. (2017). We also prove that surplus-invariant, law-invariant, and

numéraire-invariant acceptance sets must be the sets induced by VaR.

The remainder of the paper is as follows: Section 2 delivers the main results and Section 3 concludes. All proofs are placed in the appendix.

2. Main Results

2.1. Acceptance Sets

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ as the set of all proper random variables on this space. Let \mathcal{X} be a subset of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ that contains $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, the set of bounded random variables. Here, \mathcal{X} can include unbounded random variables; for example, \mathcal{X} can be $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p \in [1, +\infty)$ that represents the set of random variables with finite L^p -norm. Each element X in \mathcal{X} represents the *capital position* of a firm, i.e., the firm's assets net of its liability, at the end of a given period. Thus the positive part of X , denoted as $X^+ := \max(X, 0)$, is the *surplus* of the firm's shareholders and the negative part of X , denoted as $X^- := -\min(X, 0)$, is the *option to default* of the firm. The shareholders of the firm take the surplus but do not pay the option to default due to limited liability, therefore the liability holders cannot take the surplus but have to pay the option to default.

The regulator proposes a capital adequacy test by specifying an *acceptance set* \mathcal{A} . We make the standard assumption in the risk measure literature (see, e.g., Artzner et al. 1999, Staum 2013, Koch-Medina et al. 2015) that whether a firm passes the capital adequacy test only depends on the capital position of the firm. Hence \mathcal{A} is a subset of \mathcal{X} : a firm passes the test if and only if its capital position lies in \mathcal{A} . We introduce the following properties for the acceptance set \mathcal{A} :

(i) *Surplus invariance*: for any $X, Y \in \mathcal{X}$, if $X \in \mathcal{A}$ and $X^- \geq Y^-$ almost surely (a.s.), then $Y \in \mathcal{A}$.

(ii) *Law invariance*: for any $X, Y \in \mathcal{X}$, if $X \in \mathcal{A}$ and Y have the same distribution as X , then $Y \in \mathcal{A}$.

(iii) *Conicity*: for any $X \in \mathcal{X}$ and $\lambda > 0$ such that $\lambda X \in \mathcal{X}$, if $X \in \mathcal{A}$, then $\lambda X \in \mathcal{A}$.

(iv) *Numéraire invariance*: for any $X \in \mathcal{X}$ and any strictly positive random variable Z on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $ZX \in \mathcal{X}$, if $X \in \mathcal{A}$, then $ZX \in \mathcal{A}$.

(v) *Truncation closedness*: for any element $X \in \mathcal{X}$, if $\min(\max(-d, X), d) \in \mathcal{A}$ for any $d > 0$, then $X \in \mathcal{A}$.

The surplus-invariance property is proposed by Koch-Medina et al. (2015),¹ extending the excess-invariance property of the shortfall risk measures and the loss-dependence property of the loss-based risk measures proposed by Staum (2013), Cont et al. (2013), respectively.² The surplus-invariance property stipulates that if firm A passes the test, then firm B whose option to default is smaller than that of firm A should also pass the test. This property is satisfied by the acceptance set that is associated with VaR, i.e., $\mathcal{A} = \{X \in \mathcal{X} |$

$V@R_\alpha(X) \leq 0\}$, and the one that is associated with a shortfall risk measure, i.e., $\mathcal{A} = \{X \in \mathcal{X} | \mathbb{E}[l(X^-)] \leq c\}$, where l is a nonconstant and increasing function. See Staum (2013), Koch-Medina et al. (2017) for more examples and discussion.

If an acceptance set \mathcal{A} satisfies the surplus-invariance property, then \mathcal{A} satisfies the property that, for any $X, Y \in \mathcal{X}$, if $X \leq Y$ a.s. and $X \in \mathcal{A}$, then $Y \in \mathcal{A}$. In fact, the latter property is used in the definition of acceptance sets in Koch-Medina et al. (2015, 2017).

Law invariance is important for the regulator to use historical data to backtest the models used by firms in conducting the capital adequacy test. Conicity stipulates that scaling the capital position of a firm by a positive constant does not change the acceptability of the firm. This property is assumed in coherent risk measures (Artzner et al. 1999), but is not required in convex risk measures (Föllmer and Schied 2002, Frittelli and Rosazza Gianin 2002). Weber et al. (2013) show that coherent risk measures applied to systems that exhibit price impact may induce convex risk measures that are not conic.

Numéraire invariance is introduced in Artzner et al. (2009) and further investigated in Koch-Medina et al. (2017). It means that “the acceptance set should not depend on the choice of the eligible (numéraire) asset” (Artzner et al. 2009, p. 114). Numéraire invariance can accommodate the situation that the assets and liabilities of firms may be denominated in different currencies. Koch-Medina and Munari (2016, Section 6) show that the acceptance set associated with VaR is numéraire invariant but that associated with expected shortfall is not. Clearly, numéraire invariance implies conicity.³ Without assuming numéraire invariance, the currency risk can also be explicitly incorporated in capital adequacy tests by using vector-valued risk measures; see, e.g., Jouini et al. (2004).

The truncation-closedness property simply means that if any truncated version of the (possibly unbounded) capital position X is acceptable, then X itself is also acceptable. This property is similar to the continuity axiom for the distortion risk measure ρ , which postulates that ρ satisfies

$$\lim_{d \rightarrow +\infty} \rho(\min(d, X)) = \lim_{d \rightarrow +\infty} \rho(\max(-d, X)) = \rho(X)$$

(Wang et al. 1997). The truncation-closedness property automatically holds for any acceptance set if $\mathcal{X} = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and is satisfied by some commonly used acceptance sets. For example, for any distortion risk measure

$$\rho_h(X) := \int_0^\infty h(\mathbb{P}(X > x)) dx + \int_{-\infty}^0 (h(\mathbb{P}(X > x)) - 1) dx,$$

where h is a distortion function (h is increasing with $h(0) = 0$ and $h(1) = 1$), $\mathcal{A}_{\rho_h} = \{X \in \mathcal{X} | \rho_h(X) \leq 0\}$ is truncation closed because $\lim_{d \rightarrow \infty} \rho_h(\min(\max(-d, X), d)) = \rho_h(X)$.

2.2. Value at Risk

The current practice of capital adequacy tests in the Basel II Accord and the Solvency II is to use VaR; the Basel III Accord imposes that starting from January 1, 2018, the expected shortfall at 97.5% level under stressed scenarios will be used in the test of capital adequacy for market risk (Basel Committee on Banking Supervision 2016). To define VaR formally, we introduce some notations. For a given random variable X , denote F_X as its (right-continuous) cumulative distribution function. Denote F_X^{-1} as the left-continuous quantile function of X , i.e.,

$$F_X^{-1}(t) := \sup\{x \in \mathbb{R} \mid F_X(x) < t\}$$

$$= \inf\{x \in \mathbb{R} \mid F_X(x) \geq t\}, \quad t \in [0, 1].$$

In particular, $F_X^{-1}(0) = -\infty$ and $F_X^{-1}(1) = \text{ess sup } X := \inf\{x \mid \mathbb{P}(X \leq x) = 1\}$ by definition. Furthermore, $F_X^{-1}(t+) = \inf\{x \in \mathbb{R} \mid F_X(x) > t\}$ is the right-continuous quantile function of X (Föllmer and Schied 2016, Lemma A.19); in particular, $F_X^{-1}(0+) = \lim_{z \downarrow 0} F_X^{-1}(z) = \text{ess inf } X := \sup\{x \mid \mathbb{P}(X \leq x) = 0\}$, and $F_X^{-1}(1+) = +\infty$. It is well known that for any $x \in \mathbb{R}$ and $t \in [0, 1]$, (i) $F_X(x) < t$ if and only if $F_X^{-1}(t) > x$ and (ii) $F_X(x-) \leq t$ if and only if $F_X^{-1}(t+) \geq x$, where $F_X(x-) := \lim_{z \uparrow x} F_X(z)$. It is easy to show that⁴ for any continuous, increasing function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$, $F_{\varphi(X)}^{-1}(z) = \varphi(F_X^{-1}(z))$, $z \in (0, 1)$ and for any continuous decreasing function $\psi: \mathbb{R} \rightarrow \mathbb{R}$, $F_{\psi(X)}^{-1}(z) = \psi(F_X^{-1}((1-z)+))$, $z \in (0, 1)$.

VaR of X at level α is defined as the quantile function of $-X$ at level α . Formally, for any $\alpha \in [0, 1]$, we define the lower α -VaR and upper α -VaR of X , respectively, to be

$$V\@R_\alpha^l(X) := F_{-X}^{-1}(\alpha) = -F_X^{-1}((1-\alpha)+), \quad (1)$$

$$V\@R_\alpha^u(X) := F_{-X}^{-1}(\alpha+) = -F_X^{-1}(1-\alpha). \quad (2)$$

For each $\alpha \in [0, 1]$, we define the following three acceptance sets based on the default probability of the firm:

$$\mathcal{A}_\alpha^- := \{X \in \mathcal{X} \mid \mathbb{P}(X < 0) < 1 - \alpha\}, \quad (3)$$

$$\mathcal{A}_\alpha^0 := \{X \in \mathcal{X} \mid \mathbb{P}(X \leq -\epsilon) < 1 - \alpha, \text{ for any } \epsilon > 0\}, \quad (4)$$

$$\mathcal{A}_\alpha^+ := \{X \in \mathcal{X} \mid \mathbb{P}(X < 0) \leq 1 - \alpha\}. \quad (5)$$

In the Basel II Accord, a capital position X is acceptable if and only if its VaR at certain level α (e.g., 99%) is less than or equal to zero. The following Proposition 1 shows that \mathcal{A}_α^- , \mathcal{A}_α^0 , and \mathcal{A}_α^+ are indeed induced by VaR. Moreover, these acceptance sets are surplus invariant, law invariant, conic, and truncation closed; furthermore, \mathcal{A}_α^- and \mathcal{A}_α^+ are numéraire invariant.

Proposition 1. (i) For any $\alpha \in [0, 1]$, \mathcal{A}_α^- , \mathcal{A}_α^0 , and \mathcal{A}_α^+ satisfy

$$\mathcal{A}_\alpha^- = \{X \in \mathcal{X} \mid V\@R_{\alpha+\delta}^u(X) \leq 0, \text{ for some } \delta \in (0, 1-\alpha)\}, \quad (6)$$

$$\mathcal{A}_\alpha^0 = \{X \in \mathcal{X} \mid V\@R_\alpha^u(X) \leq 0\}, \quad (7)$$

$$\mathcal{A}_\alpha^+ = \{X \in \mathcal{X} \mid V\@R_\alpha^l(X) \leq 0\}. \quad (8)$$

In particular, $\mathcal{A}_1^- = \mathcal{A}_1^0 = \emptyset$ and $\mathcal{A}_1^+ = \{X \in \mathcal{X} \mid X \geq 0 \text{ a.s.}\}$.

(ii) For any $\alpha \in [0, 1]$, \mathcal{A}_α^- , \mathcal{A}_α^0 , and \mathcal{A}_α^+ are surplus-invariant, law-invariant, conic, and truncation-closed acceptance sets, and $\mathcal{A}_\alpha^- \subseteq \mathcal{A}_\alpha^0 \subseteq \mathcal{A}_\alpha^+$; moreover, \mathcal{A}_α^- and \mathcal{A}_α^+ are numéraire invariant. For any $0 \leq \alpha_1 < \alpha_2 \leq 1$, $\mathcal{A}_{\alpha_1}^- \supseteq \mathcal{A}_{\alpha_2}^-$.

Note that the surplus invariance of \mathcal{A}_α^+ is already shown in Staum (2013, Example 4.1) and Koch-Medina et al. (2017, Example 1). It is also worth noting that the acceptance set \mathcal{A}_1^+ requires that a firm is acceptable if and only if it never defaults, therefore, it is only formally a VaR-induced acceptance set. Finally, the differences between \mathcal{A}_α^- , \mathcal{A}_α^0 , and \mathcal{A}_α^+ are minor, although, in general, $\mathcal{A}_\alpha^- \subsetneq \mathcal{A}_\alpha^0$ and $\mathcal{A}_\alpha^0 \subsetneq \mathcal{A}_\alpha^+$. Moreover, \mathcal{A}_α^0 is not numéraire invariant, in general, as shown by Example 1 in Appendix D.

2.3. Surplus-Invariant, Law-Invariant, Conic, and Truncation-Closed Acceptance Sets Must Be the Sets Induced by VaR

We prove that any surplus-invariant, law-invariant, conic (or numéraire-invariant), and truncation-closed acceptance set must be a set induced by VaR. To this end, we follow the literature of law-invariant risk measures (e.g., Kusuoka 2001, Jouini et al. 2006, Frittelli and Rosazza Gianin 2005) to restrict ourselves to atomless probability spaces that support a uniform random variable. Our results also hold for equiprobable probability spaces (($\Omega, \mathcal{F}, \mathbb{P}$) is an equiprobable probability space if $\Omega = \{\omega_1, \dots, \omega_n\}$ and $\mathbb{P}(\omega_i) = 1/n, \forall i$); see Appendix C.

Theorem 1. Assume ($\Omega, \mathcal{F}, \mathbb{P}$) is atomless. Then,

(i) If \mathcal{A} is a surplus-invariant, law-invariant, conic, and truncation-closed acceptance set, then there exists $\alpha \in [0, 1]$ such that either $\mathcal{A}_\alpha^- \subseteq \mathcal{A} \subseteq \mathcal{A}_\alpha^0$ or $\mathcal{A} = \mathcal{A}_\alpha^+$.

(ii) \mathcal{A} is a surplus-invariant, law-invariant, numéraire-invariant, and truncation-closed acceptance set if and only if there exists $\alpha \in [0, 1]$ such that $\mathcal{A} = \mathcal{A}_\alpha^-$ or $\mathcal{A} = \mathcal{A}_\alpha^+$.

Theorem 1 shows that surplus-invariant, law-invariant, conic (or numéraire-invariant), and truncation-closed acceptance sets are essentially the sets induced by VaR. As a technical comment, there exists a surplus-invariant, law-invariant, conic, and truncation-closed acceptance set \mathcal{A} such that $\mathcal{A}_\alpha^- \subsetneq \mathcal{A} \subsetneq \mathcal{A}_\alpha^0$; see Example 2 in Appendix D.⁵

It is worth noting that an acceptance set \mathcal{A} that satisfies the properties in Theorem 1 cannot be induced by expected shortfall unless \mathcal{A} is trivial, i.e., is the empty set or the whole set of capital positions in \mathcal{X} whose expected shortfall is well defined (i.e., finite). In fact, for the sake of contradiction, suppose such a nontrivial \mathcal{A} is induced by the expected shortfall at certain level $\beta \in [0, 1]$; i.e., $\mathcal{A} = \{X \in \mathcal{X} \mid \text{ES}_\beta(X) \in B\}$, where $B \subseteq \mathbb{R}$ and $\text{ES}_\beta(X) := (1/(1-\beta)) \int_\beta^1 V\@R_\alpha^l(X) d\alpha = (1/(1-\beta)) \cdot \int_\beta^1 F_X^{-1}(\alpha) d\alpha$ for $\beta \in [0, 1]$ and $\text{ES}_1(X) := \text{ess sup}(-X)$. Note that $\text{ES}_\beta(X)$ is well defined for X whose negative part is integrable when $\beta \in (0, 1)$, for X that is integrable when $\beta = 0$, and for X that is bounded from below

when $\beta = 1$. Now, for any constant $c \geq 0$ and $X \in \mathcal{X}$, we have $c^- = 0 \leq X^-$. Recalling that \mathcal{A} is nonempty and surplus invariant, we must have $c \in \mathcal{A}$, implying that $(-\infty, 0] \subseteq B$. In contrast, if there exists $X \in \mathcal{A}$ such that $ES_\beta(X) > 0$, then it follows from the concavity of \mathcal{A} that $(0, \infty) \subseteq B$, which implies that $B = \mathbb{R}$ and thus $\mathcal{A} = \{X \in \mathcal{X} \mid ES_\beta(X) \in \mathbb{R}\}$, a trivial set, which leads to a contradiction. Hence there does not exist $X \in \mathcal{A}$ such that $ES_\beta(X) > 0$, implying $B = (-\infty, 0]$; i.e., $\mathcal{A} = \{X \in \mathcal{X} \mid ES_\beta(X) \leq 0\}$. And then, it follows from Koch-Medina et al. (2017, Example 5) that $\{X \in \mathcal{X} \mid ES_\beta(X) \leq 0\}$ is not surplus invariant, which contradicts the assumption that \mathcal{A} is surplus invariant. In addition, even the trivial set $\mathcal{A} = \{X \in \mathcal{X} \mid ES_\beta(X) \in \mathbb{R}\}$ may not satisfy the properties in Theorem 1. For example, if $\mathcal{X} = \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, then $\mathcal{A} = \{X \in \mathcal{X} \mid ES_\beta(X) \in \mathbb{R}\}$ is not truncation closed.

Theorem 1 can be simplified if we further require that an acceptance set is closed with respect to some suitable topology on \mathcal{X} . In particular, the space $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ is a complete metric space under the Ky Fan metric (the metric is defined as $d(X, Y) := \mathbb{E}[\min(|X - Y|, 1)]$, $\forall X, Y$), which metrizes convergence in probability; see, e.g., Dudley (2002, Theorems 9.2.2, 9.2.3, pp. 289–290). Therefore, a subset of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ is closed under the Ky Fan metric topology if and only if it is closed under convergence in probability. Hence we consider the following topological property of the acceptance set \mathcal{A} :

(vi) *Convergence-in-probability-closedness*: if $X_n \rightarrow X$ in \mathcal{X} in probability as $n \rightarrow \infty$ and $X_n \in \mathcal{A}$ for all n , then $X \in \mathcal{A}$.

Since $\min(\max(-d, X), d) \rightarrow X$ a.s. as $d \rightarrow \infty$, it follows that convergence-in-probability-closedness implies truncation closedness. Theorem 1 can be simplified to be the following Theorem 2 if we require that an acceptance set is closed with respect to the metric topology of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, or, equivalently, closed with respect to convergence in probability.

Theorem 2. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless. Then, \mathcal{A} is a nonempty, surplus-invariant, law-invariant, conic (or numéraire-invariant), and convergence-in-probability-closed acceptance set if and only if there exists $\alpha \in [0, 1]$ such that $\mathcal{A} = \mathcal{A}_\alpha^+$.

The proof of Theorem 2 is obtained by applying Theorem 1 and verifying that, for $\alpha \in [0, 1]$, \mathcal{A}_α^+ is the closure of \mathcal{A}_α^- in \mathcal{X} under the metric topology of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$. Similar results can be obtained if one considers $\mathcal{X} \subseteq \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p \in [1, +\infty)$ and the closedness with respect to the L^p -norm, but not for the case of $\mathcal{X} = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and the closedness with respect to the L^∞ -norm or the weak-star topology; see Appendix B.

3. Conclusions

The regulator is interested in proposing a capital adequacy test by specifying an acceptance set for firms' capital positions at the end of a given period. Such test aims

to protect firms' liability holders, therefore whether a firm passes the test should not depend on the surplus of the firm's shareholders, a property referred to as surplus invariance. We prove that surplus-invariant, law-invariant, and conic acceptance sets must be the sets induced by VaR, i.e., must be the set of capital positions whose VaR at a given level is less than or equal to zero. We also prove that surplus-invariant, law-invariant, and numéraire-invariant acceptance sets must be the sets induced by VaR.

Our results contribute to show the potential conflicts between competing properties of capital adequacy tests. Koch-Medina et al. (2015) show that the class of acceptance sets that are convex and numéraire invariant (or equivalently, coherent and surplus invariant) is limited to scenario tests; in addition, law invariance further narrows the domain to the positive cone alone. We show that, if one drops convexity, the choice is enlarged only by VaR-induced acceptance sets. In contrast, if one drops concavity or numéraire invariance, then there is a rich class of surplus-invariant, law-invariant, and convex acceptance sets; see, for instance, Cont et al. (2013); Koch-Medina et al. (2015, 2017). As a result, if one finds the range of choices of acceptance sets too restrictive, then one needs to sacrifice one of the properties under investigation and is therefore forced to setup criteria to prioritize competing properties such as convexity, coherence, surplus invariance, numéraire invariance, and law invariance. These properties might be desirable if considered alone but lead to a narrow range of choices when put together.⁶

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Appendix A. Proofs

Proof of Proposition 1. (i) For $\alpha = 1$, it follows from the definitions in (3)–(5) that $\mathcal{A}_1^- = \mathcal{A}_1^0 = \emptyset$, and $\mathcal{A}_1^+ = \{X \in \mathcal{X} \mid X \geq 0 \text{ a.s.}\}$, which in combination with $V@R_1^u(X) = -F_X^{-1}(0) = +\infty$ and $V@R_1^l(X) = -F_X^{-1}(0+) = -\text{ess inf } X$, imply that (6)–(8) hold for $\alpha = 1$.

We then prove (6)–(8) for $\alpha \in [0, 1)$. Recall that for any random variable X , $x \in \mathbb{R}$, and $t \in [0, 1]$, $F_X(x) < t \Leftrightarrow F_X^{-1}(t) > x$ and $F_X(x) \leq t \Leftrightarrow F_X^{-1}(t+) \geq x$. We will show that $F_X^{-1}(1 - \alpha - \delta) \geq 0$ for some $\delta \in (0, 1 - \alpha) \Leftrightarrow F_X^{-1}((1 - \alpha - \delta)+) \geq 0$ for some $\delta' \in (0, 1 - \alpha)$. The “ \Rightarrow ” direction follows from $F_X^{-1}((1 - \alpha - \delta)+) \geq F_X^{-1}(1 - \alpha - \delta)$. To show the “ \Leftarrow ” direction, suppose $F_X^{-1}((1 - \alpha - \delta')+) \geq 0$ for some $\delta' \in (0, 1 - \alpha)$. Let $\delta = \delta'/2$. Then, the monotonicity of $F_X^{-1}(\cdot)$ implies that $F_X^{-1}(1 - \alpha - \delta) \geq F_X^{-1}((1 - \alpha - \delta')+) \geq 0$.

Then, by (2), $V@R_{\alpha+\delta}^u(X) \leq 0$ for some $\delta \in (0, 1 - \alpha) \Leftrightarrow F_X^{-1}(1 - (\alpha + \delta)) \geq 0$ for some $\delta \in (0, 1 - \alpha) \Leftrightarrow F_X^{-1}((1 - (\alpha + \delta'))+) \geq 0$ for some $\delta' \in (0, 1 - \alpha) \Leftrightarrow 1 - (\alpha + \delta') \geq F_X(0-)$ for some $\delta' \in (0, 1 - \alpha) \Leftrightarrow \mathbb{P}(X < 0) = F_X(0-) < 1 - \alpha$. Thus (6) holds. Similarly,

by (2), $V@R_\alpha^u(X) \leq 0 \Leftrightarrow F_X^{-1}(1-\alpha) \geq 0 \Leftrightarrow -\epsilon < F_X^{-1}(1-\alpha)$ for any $\epsilon > 0 \Leftrightarrow \mathbb{P}(X \leq -\epsilon) = F_X(-\epsilon) < 1-\alpha$ for any $\epsilon > 0$. Thus (7) holds. Finally, by (1), $V@R_\alpha^l(X) \leq 0 \Leftrightarrow F_X^{-1}((1-\alpha)^+) \geq 0 \Leftrightarrow \mathbb{P}(X < 0) = F_X(0-) \leq 1-\alpha$. Thus (8) holds.

(ii) By definition, $\mathcal{A}_\alpha^-, \mathcal{A}_\alpha^0$, and \mathcal{A}_α^+ are law invariant and conic. Moreover, it follows from (3) that, for any $X \in \mathcal{A}_\alpha^-$ and any $Y \in \mathcal{X}$ satisfying $Y^- \leq X^-$ a.s., $\mathbb{P}(Y < 0) = \mathbb{P}(-Y^- < 0) \leq \mathbb{P}(-X^- < 0) = \mathbb{P}(X < 0) < 1-\alpha$, which implies $Y \in \mathcal{A}_\alpha^-$. Hence \mathcal{A}_α^- is surplus invariant. A similar argument with the application of (4) and (5) yields that \mathcal{A}_α^0 and \mathcal{A}_α^+ are also surplus invariant. For any $X \in \mathcal{X}$, and $d > 0$, it holds that $\mathbb{P}(\min(\max(-d, X), d) < 0) = \mathbb{P}(X < 0)$, which in combination with (3) implies that \mathcal{A}_α^- is truncation closed. A similar argument yields that \mathcal{A}_α^0 and \mathcal{A}_α^+ are also truncation closed.

Note that, for any $X \in \mathcal{X}$ and any strictly positive random variable Z such that $ZX \in \mathcal{X}$, $\mathbb{P}(ZX < 0) = \mathbb{P}(X < 0)$. Then, it follows from (3) and (5) that \mathcal{A}_α^- and \mathcal{A}_α^+ are numéraire invariant.

Finally, it follows from (3)–(5) that $\mathcal{A}_\alpha^- \subseteq \mathcal{A}_\alpha^0 \subseteq \mathcal{A}_\alpha^+$ for any $\alpha \in [0, 1]$, and $\mathcal{A}_{\alpha_1}^- \supseteq \mathcal{A}_{\alpha_2}^+$ for any $0 \leq \alpha_1 < \alpha_2 \leq 1$. \square

Proof of Theorem 1. The proof of part (i) is as follows. If $\mathcal{A} = \emptyset$, then $\mathcal{A} = \mathcal{A}_1^- = \mathcal{A}_1^0$, therefore we assume that \mathcal{A} is nonempty in the following. Define

$$\alpha := 1 - \sup_{X \in \mathcal{A}} \mathbb{P}(X < 0) = 1 - \sup_{X \in \mathcal{A}} F_X(0-). \quad (\text{A.1})$$

If $\alpha = 1$, then by definition of α , $\mathcal{A} \subseteq \{X \in \mathcal{X} \mid X \geq 0 \text{ a.s.}\} = \mathcal{A}_1^+$. In contrast, because \mathcal{A} is nonempty and surplus invariant, $\mathcal{A}_1^+ \subseteq \mathcal{A}$. Therefore $\mathcal{A} = \mathcal{A}_1^+$.

Now, suppose $\alpha \in [0, 1)$. First, we will show $\mathcal{A}_\alpha^- \subseteq \mathcal{A}$; i.e., for any $Z \in \mathcal{A}_\alpha^- \subseteq \mathcal{X}$, we will show that $Z \in \mathcal{A}$. Since \mathcal{A} is truncation closed, we only need to show that for any fixed $d > 0$, $Y := \min(\max(-d, Z), d) \in \mathcal{A}$. Since $Z \in \mathcal{A}_\alpha^-$, it follows from (3) that $\mathbb{P}(Y < 0) = \mathbb{P}(Z < 0) < 1-\alpha$. Hence there exists $\delta \in (0, 1-\alpha)$ such that $F_Y(0-) = \mathbb{P}(Y < 0) \leq 1-\alpha-\delta$. By the definition of α , there exists $X \in \mathcal{A}$ such that $F_X(0-) = \mathbb{P}(X < 0) > 1-\alpha-\delta$. Because $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, there exists a uniformly distributed random variable U on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X = F_X^{-1}(U+)$ a.s. (see, e.g., Föllmer and Schied 2016, Lemma A.32). Define $\tilde{Y} := F_Y^{-1}(U+)$. Then, it follows from Föllmer and Schied (2016, Lemma A.23) that \tilde{Y} has the same distribution as Y . Because Y and \tilde{Y} are bounded and thus contained in \mathcal{X} and \mathcal{A} is law invariant, we only need to show that $\tilde{Y} \in \mathcal{A}$.

Because $F_X(0-) > 1-\alpha-\delta$, we have $F_X^{-1}((1-\alpha-\delta)^+) < 0$. As a result, because $F_Y^{-1}(0+) \geq -d$, there exists $\epsilon > 0$ such that $\epsilon F_Y^{-1}(0+) \geq F_X^{-1}((1-\alpha-\delta)^+)$, which implies that $\epsilon F_Y^{-1}(z+) \geq \epsilon F_Y^{-1}(0+) \geq F_X^{-1}((1-\alpha-\delta)^+) \geq F_X^{-1}(z+)$ for any $z \in (0, 1-\alpha-\delta]$. Moreover, because $F_Y(0-) \leq 1-\alpha-\delta$, we have $F_Y^{-1}(z+) \geq F_Y^{-1}((1-\alpha-\delta)^+) \geq 0, z \in (1-\alpha-\delta, 1)$. Consequently, $\min(\epsilon F_Y^{-1}(z+), 0) \geq \min(F_X^{-1}(z+), 0)$ for any $z \in (0, 1)$ and thus $(\epsilon \tilde{Y})^- = -\min(\epsilon F_Y^{-1}(U+), 0) \leq -\min(F_X^{-1}(U+), 0) = X^-$ a.s. Because $\epsilon \tilde{Y} \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{X}$, $X \in \mathcal{A}$, and \mathcal{A} is surplus invariant, $\epsilon \tilde{Y} \in \mathcal{A}$. Then, because $\tilde{Y} \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{X}$ and \mathcal{A} is conic, we obtain that $\tilde{Y} \in \mathcal{A}$.

In contrast, (5) and the definition of α in (A.1) imply $\mathcal{A} \subseteq \mathcal{A}_\alpha^+$. Thus we have proved that $\mathcal{A}_\alpha^- \subseteq \mathcal{A} \subseteq \mathcal{A}_\alpha^+$. Hence there are only two cases: (1) $\mathcal{A}_\alpha^- \subseteq \mathcal{A} \subseteq \mathcal{A}_\alpha^0$ and (2) there exists $X \in \mathcal{A}$ such that $X \notin \mathcal{A}_\alpha^0$. To complete the proof, we only need to show that the second case leads to $\mathcal{A} = \mathcal{A}_\alpha^+$.

In fact, in the second case, let $X \in \mathcal{A}$ be such that $X \notin \mathcal{A}_\alpha^0$. Since $X \in \mathcal{A}$, it follows from (A.1) that $F_X(0-) \leq 1-\alpha$, which implies that $F_X^{-1}((1-\alpha)^+) \geq 0$. Since $X \notin \mathcal{A}_\alpha^0$, it follows from (7) that $V@R_\alpha^u(X) > 0$, which in combination with (2) implies that $F_X^{-1}(1-\alpha) < 0$.

Because $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, there exists a uniform random variable U such that $X = F_X^{-1}(U)$ a.s. (see, e.g., Föllmer and Schied 2016, Lemma A.32). Now, for any $Z \in \mathcal{A}_\alpha^+ \subseteq \mathcal{X}$, we will show that, $Z \in \mathcal{A}$. Since \mathcal{A} is truncation closed, we only need to show that for any fixed $d > 0$, $Y := \min(d, \max(-d, Z)) \in \mathcal{A}$. Since $Z \in \mathcal{A}_\alpha^+$, $Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{X}$, and $\mathbb{P}(Y < 0) = \mathbb{P}(Z < 0)$, it follows from (5) that $Y \in \mathcal{A}_\alpha^+$ and $F_Y(0-) \leq 1-\alpha$, which implies that $F_Y^{-1}((1-\alpha)^+) \geq 0$. Define $\tilde{Y} := F_Y^{-1}(U)$. Then, it follows from Föllmer and Schied (2016, Lemma A.23) that \tilde{Y} has the same distribution as Y . Because $F_X^{-1}(1-\alpha) < 0$ and $F_Y^{-1}(0+) \geq -d$, there exists $\epsilon > 0$ such that $\epsilon F_Y^{-1}(0+) \geq F_X^{-1}(1-\alpha)$, which implies that $\epsilon F_Y^{-1}(z) \geq \epsilon F_Y^{-1}(0+) \geq F_X^{-1}(1-\alpha) \geq F_X^{-1}(z)$, $\forall z \in (0, 1-\alpha]$. Moreover, because $F_Y^{-1}(z) \geq F_Y^{-1}((1-\alpha)^+) \geq 0, z \in (1-\alpha, 1)$, we conclude that $\min(\epsilon F_Y^{-1}(z), 0) \geq \min(F_X^{-1}(z), 0)$ for any $z \in (0, 1)$. Consequently, $(\epsilon \tilde{Y})^- = -\min(\epsilon F_Y^{-1}(U), 0) \leq -\min(F_X^{-1}(U), 0) = X^-$ a.s. Since \mathcal{A} is surplus invariant and $\epsilon \tilde{Y} \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{X}$, it follows that $\epsilon \tilde{Y} \in \mathcal{A}$. Because Y and \tilde{Y} are bounded and thus contained in \mathcal{X} , conicity of \mathcal{A} implies that $\tilde{Y} \in \mathcal{A}$ and law invariance of \mathcal{A} implies that $Y \in \mathcal{A}$. Therefore $\mathcal{A} = \mathcal{A}_\alpha^+$, which completes the proof of part (i).

The proof of part (ii) is as follows. The “if” direction follows from Proposition 1. To show the “only if” direction, suppose \mathcal{A} is surplus invariant, law invariant, numéraire invariant, and truncation closed. Since numéraire invariance implies conicity, it follows from part (i) of the theorem that there exists $\alpha \in [0, 1]$, such that $\mathcal{A}_\alpha^- \subseteq \mathcal{A} \subseteq \mathcal{A}_\alpha^0$ or $\mathcal{A} = \mathcal{A}_\alpha^+$. There are two cases: $\alpha = 1$ and $\alpha \in [0, 1)$. In the first case, it follows from (3) and (4) that $\mathcal{A}_\alpha^- = \mathcal{A}_\alpha^0 = \emptyset$. Hence $\mathcal{A} = \mathcal{A}_\alpha^- = \emptyset$ or $\mathcal{A} = \mathcal{A}_\alpha^+$. In the second case, suppose for the sake of contradiction that $\mathcal{A}_\alpha^- \subseteq \mathcal{A} \subseteq \mathcal{A}_\alpha^0$. Then, there exists $X \in \mathcal{A} \subseteq \mathcal{A}_\alpha^0$ such that $X \notin \mathcal{A}_\alpha^-$. Fix $d > 0$ and define $Y := \min(d, \max(-d, X))$. Noting $Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{X}$, and $Y^- \leq X^-$, we conclude from the surplus invariance of \mathcal{A} that $Y \in \mathcal{A} \subseteq \mathcal{A}_\alpha^0$. It follows from (3) and $\mathbb{P}(Y < 0) = \mathbb{P}(X < 0)$ that $Y \notin \mathcal{A}_\alpha^-$, and $\mathbb{P}(Y < 0) \geq 1-\alpha$. Define $Z := -(1/Y)\mathbf{1}_{\{Y < 0\}} + \mathbf{1}_{\{Y \geq 0\}}$. Then, $Z > 0$, and $ZY = -\mathbf{1}_{\{Y < 0\}} + Y\mathbf{1}_{\{Y \geq 0\}} \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{X}$. Moreover, for any $\epsilon \in (0, 1)$, $\mathbb{P}(ZY \leq -\epsilon) = \mathbb{P}(Y < 0) \geq 1-\alpha$, which implies from (4) that $ZY \notin \mathcal{A}_\alpha^0$ and thus $ZY \notin \mathcal{A}$. This contradicts the assumption that \mathcal{A} is numéraire invariant. \square

Proof of Theorem 2. To prove the “if” part, by Proposition 1, we only need to show that \mathcal{A}_α^+ is convergence-in-probability closed. Suppose $X_k \in \mathcal{A}_\alpha^+, k = 1, 2, \dots$, and $X_k \rightarrow X \in \mathcal{X}$ in probability as $k \rightarrow \infty$. Then, for any $\delta > 0$, $\mathbb{P}(X \leq -\delta) = \mathbb{P}(X \leq -\delta, |X_k - X| \geq \delta/2) + \mathbb{P}(X \leq -\delta, |X_k - X| < \delta/2) \leq \mathbb{P}(|X_k - X| \geq \delta/2) + \mathbb{P}(X_k \leq -\delta/2) \leq \mathbb{P}(|X_k - X| \geq \delta/2) + 1-\alpha$. Letting $k \rightarrow \infty$, we obtain $\mathbb{P}(X \leq -\delta) \leq 1-\alpha$. Then, letting $\delta \downarrow 0$ yields $\mathbb{P}(X < 0) \leq 1-\alpha$, which in combination with (5) implies that $X \in \mathcal{A}_\alpha^+$.

Next, we prove the “only if” part. Recall part (i) of Theorem 1 and the convergence-in-probability closedness of \mathcal{A}_α^+ in \mathcal{X} , $\alpha \in [0, 1]$. Because \mathcal{A} is assumed to be nonempty and $\mathcal{A}_1^- = \mathcal{A}_1^0 = \emptyset$, we only need to show that for any $\alpha \in [0, 1)$, the convergence-in-probability closure of \mathcal{A}_α^- in \mathcal{X} is \mathcal{A}_α^+ , i.e., for any $X \in \mathcal{A}_\alpha^+ \setminus \mathcal{A}_\alpha^-$, there exists a sequence $X_k, k = 1, 2, \dots$ such

that $X_k \in \mathcal{A}_\alpha^-, \forall k$, and $X_k \rightarrow X$ in probability as $k \rightarrow \infty$. For any $n = 1, 2, \dots$, denote $Y_n := \min(n, \max(-n, X)) \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{X}$. Because $\mathbb{P}(Y_n < 0) = \mathbb{P}(X < 0)$, we conclude from (3) and (5) that $Y_n \in \mathcal{A}_\alpha^+ \setminus \mathcal{A}_\alpha^-$. In the following, we show that for each fixed n , we can find a sequence $Z_k^n, k = 1, 2, \dots$ in \mathcal{A}_α^- such that it converges to Y_n in probability as $k \rightarrow \infty$. Then, for any $k \in \mathbb{N}$, there exists index m_k such that $d(Z_{m_k}^n, Y_n) < 1/k$, where $d(\cdot, \cdot)$ denotes the Ky Fan metric (i.e., $d(X, Y) := \mathbb{E}[\min(|X - Y|, 1)]$), $\forall X, Y$). Define the sequence $X_k := Z_{m_k}^n, k = 1, 2, \dots$. Then, $X_k \in \mathcal{A}_\alpha^-$, and X_k converges to X in probability as $k \rightarrow \infty$.

For fixed n , because $Y_n \in \mathcal{A}_\alpha^+ \setminus \mathcal{A}_\alpha^-$, we conclude from (3) and (5) that $\mathbb{P}(Y_n < 0) = 1 - \alpha > 0$. Denote $a := \text{essinf } Y_n$; then $a \in [-n, 0)$ because $\mathbb{P}(Y_n < 0) > 0$ and $Y_n \geq -n$. There are two cases: (i) $\mathbb{P}(Y_n = a) = 0$ and (ii) $\mathbb{P}(Y_n = a) > 0$. In case (i), define $Z_k^n := Y_n \mathbf{1}_{\{Y_n \geq a + 1/k\}} \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{X}, \forall k$. By the definition of a , $\mathbb{P}(Y_n < a + 1/k) > 0, \forall k$. Hence $\mathbb{P}(Z_k^n < 0) = \mathbb{P}(Y_n < 0) - \mathbb{P}(Y_n < a + 1/k) < \mathbb{P}(Y_n < 0) = 1 - \alpha, \forall k > -1/a$, which implies that $Z_k^n \in \mathcal{A}_\alpha^-, \forall k > -1/a$. In addition, $\lim_{k \rightarrow +\infty} \mathbb{P}(Y_n < a + 1/k) = \mathbb{P}(Y_n = a) = 0$, implying that $Z_k^n \rightarrow Y_n$ a.s. and thus in probability as $k \rightarrow \infty$. In case (ii), by the intermediate value theorem for atomless measures (see, e.g., Fremlin 2010, Proposition 215D, p. 46), there exists a sequence of decreasing subsets $\{Y_n = a\} \supset A_1 \supset A_2 \supset \dots$, such that $\mathbb{P}(A_k) = (1/(k+1)) \cdot \mathbb{P}(Y_n = a), \forall k$. Define $Z_k^n := Y_n \mathbf{1}_{\{A_k\}} \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{X}$. Then, $\mathbb{P}(Z_k^n < 0) = \mathbb{P}(Y_n < 0) - \mathbb{P}(A_k) < 1 - \alpha$, which implies that $Z_k^n \in \mathcal{A}_\alpha^-, \forall k$. In addition, for any $\epsilon > 0$, $\mathbb{P}(|Z_k^n - Y_n| \geq \epsilon) \leq \mathbb{P}(A_k) \rightarrow 0$ as $k \rightarrow \infty$, which implies that $Z_k^n \rightarrow Y_n$ in probability as $k \rightarrow \infty$. \square

Appendix B. Topologies on \mathcal{X}

In this section, we first show that Theorem 2 still holds if we restrict $\mathcal{X} \subseteq \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p \in [1, +\infty)$ and consider the closedness with respect to the L^p -norm.

Corollary 1. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless and $\mathcal{X} \subseteq \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p \in [1, +\infty)$. Then, \mathcal{A} is a nonempty, surplus invariant, law invariant, conic (or numéraire-invariant), and L^p closed (i.e., \mathcal{A} is closed in \mathcal{X} under the L^p -norm) acceptance set if and only if there exists $\alpha \in [0, 1]$ such that $\mathcal{A} = \mathcal{A}_\alpha^+$.

Proof. Following the same steps in the proof of Theorem 2, we only need to show that (i) for any $\alpha \in [0, 1]$, \mathcal{A}_α^+ is closed in \mathcal{X} under the L^p -norm and (ii) for any $\alpha \in [0, 1)$, the closure of \mathcal{A}_α^- in \mathcal{X} under the L^p -norm is \mathcal{A}_α^+ . Claim (i) is true because we already showed in the proof of Theorem 2 that \mathcal{A}_α^+ is closed in \mathcal{X} under the convergence in probability, and the topology induced by the L^p -norm is stronger than that induced by the convergence in probability. To show claim (ii), recall the sequence of random variables Z_k^n in \mathcal{A}_α^- for large enough k that are constructed in the proof of Theorem 2 to converge in probability to a given bounded random variable Y_n in \mathcal{A}_α^+ . Careful examination shows that Z_k^n converges, as $k \rightarrow +\infty$, to Y_n in L^p -norm as well. Thus, claim (ii) is true. \square

Theorem 2 does not hold if we consider $\mathcal{X} = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and the closedness with respect to the L^∞ -norm because the closure of \mathcal{A}_α^- under this norm is a strict subset of \mathcal{A}_α^+ , in general. Indeed, similar to the proof of Theorem 2, we can show that \mathcal{A}_α^+ is closed in \mathcal{X} under the L^∞ -norm. Now, fix $\alpha \in [0, 1)$ and consider $X \in \mathcal{A}_\alpha^+$ such that $\mathbb{P}(X = 1) = \alpha$ and

$\mathbb{P}(X = -1) = 1 - \alpha$. For any $Y \in \mathcal{A}_\alpha^-$, we have $\mathbb{P}(Y < 0) < 1 - \alpha$, therefore $\mathbb{P}(Y \geq 0, X = -1) \geq \mathbb{P}(X = -1) - \mathbb{P}(Y < 0) > 0$. Consequently, the L^∞ -norm of $X - Y$ must be larger than or equal to 1, implying that X is not in the closure of \mathcal{A}_α^- under the L^∞ -norm.

Theorem 2 does not hold if we consider $\mathcal{X} = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and the closedness with respect to the weak-star topology because \mathcal{A}_α^+ is not closed under this topology, in general. Indeed, consider $\Omega = [0, 1], \mathcal{F}$ to be the Borel σ -algebra, and \mathbb{P} to be the Lebesgue measure. We show that for any $\alpha \in (0, 1)$, \mathcal{A}_α^+ is not closed under the weak-star topology. For any $X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, any $\epsilon > 0$, and any finite subset A of $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, denote $N(X, A, \epsilon) := \{Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \mid |\mathbb{E}[YZ] - \mathbb{E}[XZ]| < \epsilon, \forall Z \in A\}$. Then, $\{N(X, A, \epsilon) \mid X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}), A \text{ is a finite subset of } \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}), \epsilon > 0\}$ form a base of the weak-star topology; see Dunford and Schwartz (1988, Definition V.3.2). Now, consider $X^* \equiv -(1 - \alpha) \in \mathcal{X} \setminus \mathcal{A}_\alpha^+$. In the following, we show that for any finite subset A of $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and any $\epsilon > 0$, there exists $X' \in N(X^*, A, \epsilon) \cap \mathcal{A}_\alpha^+$; consequently, \mathcal{A}_α^+ is not closed under the weak-star topology.

For each $i \in \{1, \dots, n\}$, because $Z_i \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and the set of continuous functions is dense in $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, there exists a continuous function $\tilde{Z}_i(t), t \in [0, 1]$ such that $\mathbb{E}[|Z_i - \tilde{Z}_i|] < \epsilon/3$. Because \tilde{Z}_i 's are uniformly continuous, there exists an integer $m \geq 1$ such that $|\tilde{Z}_i(t) - \tilde{Z}_i(s)| < \epsilon/(6\alpha(1 - \alpha))$ for any $i \in \{1, \dots, n\}$ and any $s, t \in [0, 1]$ such that $|s - t| \leq 1/m$. Now, define

$$X_m(t) = \begin{cases} -1, & t \in \left[\frac{k-1}{m}, \frac{k-1}{m} + \frac{1-\alpha}{m} \right), \\ 0, & t \in \left[\frac{k-1}{m} + \frac{1-\alpha}{m}, \frac{k}{m} \right), \end{cases} \quad k = 1, \dots, m.$$

Then, $\mathbb{P}(X_m < 0) = \mathbb{P}(X_m = -1) = 1 - \alpha$, therefore $X_m \in \mathcal{A}_\alpha^+$. We have

$$\begin{aligned} & |\mathbb{E}[\tilde{Z}_i X_m] - \mathbb{E}[\tilde{Z}_i X^*]| \\ &= \left| \int_0^1 \tilde{Z}_i(t) X_m(t) dt - \int_0^1 \tilde{Z}_i(t) X^*(t) dt \right| \\ &= \left| \sum_{k=1}^m \left[- \int_{(k-1)/m}^{(k-1)/m + (1-\alpha)/m} \tilde{Z}_i(t) dt \right] + (1-\alpha) \int_0^1 \tilde{Z}_i(t) dt \right| \\ &= \left| \sum_{k=1}^m \left[-\alpha \int_{(k-1)/m}^{(k-1)/m + (1-\alpha)/m} \tilde{Z}_i(t) dt \right. \right. \\ &\quad \left. \left. + (1-\alpha) \int_{(k-1)/m + (1-\alpha)/m}^{k/m} \tilde{Z}_i(t) dt \right] \right| \\ &= \left| \sum_{k=1}^m \left[-\alpha \int_{(k-1)/m}^{(k-1)/m + (1-\alpha)/m} \left(\tilde{Z}_i(t) - \tilde{Z}_i \left(\frac{k-1}{m} + \frac{1-\alpha}{m} \right) \right) dt \right. \right. \\ &\quad \left. \left. + (1-\alpha) \int_{(k-1)/m + (1-\alpha)/m}^{k/m} \left(\tilde{Z}_i(t) - \tilde{Z}_i \left(\frac{k-1}{m} + \frac{1-\alpha}{m} \right) \right) dt \right] \right| \\ &\leq \sum_{k=1}^m \left[\alpha \int_{(k-1)/m}^{(k-1)/m + (1-\alpha)/m} \left| \tilde{Z}_i(t) - \tilde{Z}_i \left(\frac{k-1}{m} + \frac{1-\alpha}{m} \right) \right| dt \right. \\ &\quad \left. + (1-\alpha) \int_{(k-1)/m + (1-\alpha)/m}^{k/m} \left| \tilde{Z}_i(t) - \tilde{Z}_i \left(\frac{k-1}{m} + \frac{1-\alpha}{m} \right) \right| dt \right] \\ &< \epsilon/3, \end{aligned}$$

where the last inequality holds because $|\tilde{Z}_i(t) - \tilde{Z}_i(s)| < \epsilon/(6\alpha(1 - \alpha))$ for any $s, t \in [0, 1]$ such that $|s - t| < 1/m$. As a result,

$$\begin{aligned} & |\mathbb{E}[Z_i X_m] - \mathbb{E}[Z_i X^*]| \\ & \leq |\mathbb{E}[\tilde{Z}_i X_m] - \mathbb{E}[\tilde{Z}_i X^*]| + |\mathbb{E}[\tilde{Z}_i X_m] - \mathbb{E}[Z_i X_m]| \\ & \quad + |\mathbb{E}[\tilde{Z}_i X^*] - \mathbb{E}[Z_i X^*]| \\ & < \epsilon/3 + 2\mathbb{E}[|\tilde{Z}_i - Z_i|] < \epsilon, \end{aligned}$$

where the second inequality holds because $|X_m| \leq 1$ and $|X^*| \leq 1$, and the third inequality is the case because $\mathbb{E}[|\tilde{Z}_i - Z_i|] < \epsilon/3$. Thus $X_m \in N(X^*, A, \epsilon) \subseteq N(X, A, \epsilon)$. In contrast, $X_m \in \mathcal{A}_\alpha^+$. Therefore \mathcal{A}_α^+ is not closed under the weak-star topology.

The previous example is due to Marcel Nutz in a private communication between him and the first author of the present paper.

Appendix C. The Case of Equiprobable Probability Spaces

Theorem 3. *Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is an equiprobable probability space. Then, \mathcal{A} is a surplus-invariant, law-invariant, and conic (or numéraire-invariant) acceptance set if and only if it is the empty set or \mathcal{A}_α^+ for some $\alpha \in [0, 1]$.*

Proof of Theorem 3. Suppose $\Omega = \{\omega_1, \dots, \omega_n\}$. Then, by definition, $\mathbb{P}(\{\omega_i\}) = 1/n, i = 1, \dots, n$. First, we show that, for any random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$, there exist random variables U and \tilde{U} on $(\Omega, \mathcal{F}, \mathbb{P})$ such that (i) $X = F_X^{-1}(U+) = F_X^{-1}(\tilde{U})$ and (ii) any random variable Y on $(\Omega, \mathcal{F}, \mathbb{P})$ has the same distribution as $F_Y^{-1}(U+)$ and $F_Y^{-1}(\tilde{U})$. In fact, without loss of generality (by relabeling the states if necessary), suppose that $X(\omega_i) = x_k, n_{k-1} < i \leq n_k, k = 1, \dots, m$, where $n_0 = 0 < n_1 < \dots < n_m = n$, and $x_1 < x_2 < \dots < x_m$. Then, we define two random variables \tilde{U} and U as follows: $\tilde{U}(\omega_i) := i/n, U(\omega_i) := (i - 1)/n, i = 1, \dots, n$. Then, we have $F_X^{-1}(\tilde{U}(\omega_i)) = F_X^{-1}(i/n) = x_k = X(\omega_i)$ for any $n_{k-1} < i \leq n_k$; hence $F_X^{-1}(\tilde{U}) = X$. In addition, $F_X^{-1}(U(\omega_i)+) = F_X^{-1}(((i - 1)/n)+) = x_k = X(\omega_i)$ for any $n_{k-1} < i \leq n_k$; hence $F_X^{-1}(U+) = X$. For any random variable Y on $(\Omega, \mathcal{F}, \mathbb{P})$, suppose that $\{Y(\omega) | \omega \in \Omega\} = \{y_1, \dots, y_m\}$, where $y_1 < \dots < y_m$, and $\mathbb{P}(Y \leq y_k) = \bar{n}_k/n, k = 1, \dots, m$, where $\bar{n}_0 = 0 < \bar{n}_1 < \dots < \bar{n}_m = n$. Define a random variable \tilde{Y} as $\tilde{Y}(\omega_i) = y_k$ for any $\bar{n}_{k-1} < i \leq \bar{n}_k$. Then, we have $F_Y^{-1}(\tilde{U}(\omega_i)) = F_Y^{-1}(i/n) = y_k = \tilde{Y}(\omega_i)$ for any $\bar{n}_{k-1} < i \leq \bar{n}_k$; hence $F_Y^{-1}(\tilde{U}) = \tilde{Y}$. In addition, $F_Y^{-1}(U(\omega_i)+) = F_Y^{-1}(((i - 1)/n)+) = y_k = \tilde{Y}(\omega_i)$ for any $\bar{n}_{k-1} < i \leq \bar{n}_k$; hence $F_Y^{-1}(U+) = \tilde{Y}$. Since \tilde{Y} has the same distribution as Y , it follows that (ii) holds.

Because Ω is a finite set, all random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ are bounded and thus form the space of capital positions \mathcal{X} ; in consequence, any acceptance set is truncation closed. Then, using U and \tilde{U} as constructed previously, it can be easily verified that the proof for part (i) of Theorem 1 can also go through for the case of equiprobable probability spaces; therefore, if \mathcal{A} is a surplus-invariant, law-invariant, and conic acceptance set, then there exists $\alpha \in [0, 1]$ such that $\mathcal{A}_\alpha^- \subseteq \mathcal{A} \subseteq \mathcal{A}_\alpha^0$ or $\mathcal{A} = \mathcal{A}_\alpha^+$. When such $\alpha = 1$, we have $\mathcal{A}_1^- = \mathcal{A}_1^0 = \emptyset$, therefore $\mathcal{A} = \emptyset$ or $\mathcal{A} = \mathcal{A}_1^+$ in this case. When such $\alpha \in [0, 1)$, we claim that $\mathcal{A}_\alpha^- = \mathcal{A}_\alpha^0 = \mathcal{A}_{\alpha'}^+$ for some $\alpha' \in (\alpha, 1]$. Indeed, for any $X \in \mathcal{A}_\alpha^0$, there are two cases: (i) $X \geq 0$. In this case, by definition of \mathcal{A}_α^- , $X \in \mathcal{A}_\alpha^-$ and (ii) there exists ω such that $X(\omega) < 0$. Let $x^* := \max\{X(\omega) | X(\omega) < 0, \omega \in \Omega\}$. Note that Ω is a finite

set, therefore $x^* < 0$. Then, it follows from (4) that $\mathbb{P}(X \leq x^*) < 1 - \alpha$, which implies that $\mathbb{P}(X < 0) = \mathbb{P}(X \leq x^*) < 1 - \alpha$. By (3), we conclude $X \in \mathcal{A}_\alpha^-$. Hence $\mathcal{A}_\alpha^- = \mathcal{A}_\alpha^0$. Now, noting that the probability of any event A on $(\Omega, \mathcal{F}, \mathbb{P})$ is a multiple of $1/n$, $\mathbb{P}(A) < 1 - \alpha$ if and only if $\mathbb{P}(A) \leq 1 - \alpha'$, where $\alpha' := 1 - ([n(1 - \alpha)] - 1)/n$ and $\lceil x \rceil$ stands for the ceiling of x (i.e., the smallest integer dominating x); moreover, because $[n(1 - \alpha)] - 1 < n(1 - \alpha)$, and $\alpha < 1$, we conclude that $\alpha' \in (\alpha, 1]$. In consequence, by (3) and (5), $X \in \mathcal{A}_\alpha^-$ if and only if $X \in \mathcal{A}_{\alpha'}^+$; i.e., $\mathcal{A}_\alpha^- = \mathcal{A}_{\alpha'}^+$.

Finally, combining the previous discussion for the case $\alpha = 1$ and for the case $\alpha \in [0, 1)$, we conclude that \mathcal{A} is a surplus-invariant, law-invariant, and conic acceptance set if and only if it is the empty set or \mathcal{A}_α^+ for some $\alpha \in [0, 1]$. The previous argument still holds when conicity is replaced by numéraire invariance. \square

Appendix D. Two Examples

Example 1. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless and let $\mathcal{X} = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Consider a uniformly distributed random variable Y with support $(-1 + \alpha, \alpha)$. It follows from (3) and (4) that $Y \in \mathcal{A}_\alpha^0$ and $Y \notin \mathcal{A}_\alpha^-$. Define $Z := -(1/Y)\mathbf{1}_{\{Y < 0\}} + \mathbf{1}_{\{Y \geq 0\}}$. Then, $Z > 0$, and $ZY = -\mathbf{1}_{\{Y < 0\}} + Y\mathbf{1}_{\{Y \geq 0\}} \in \mathcal{X}$. Moreover, for any $\epsilon \in (0, 1)$, $\mathbb{P}(ZY \leq -\epsilon) = \mathbb{P}(Y < 0) = 1 - \alpha$, which in combination with (4) implies that $ZY \notin \mathcal{A}_\alpha^0$. Therefore \mathcal{A}_α^0 is not numéraire invariant.

Example 2. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless and let $\mathcal{X} = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Fix $\alpha \in (0, 1)$ and consider $Y \in \mathcal{X}$ such that Y has a uniform distribution on $(-1 + \alpha, \alpha)$. Then, $F_Y^{-1}(z) = z - (1 - \alpha), z \in (0, 1)$. Define $\mathcal{X}_1 := \{X \in \mathcal{X} | \exists \delta > 0 \text{ such that } \delta \min(F_X^{-1}(z), 0) \geq \min(F_Y^{-1}(z), 0), \forall z \in (0, 1)\}$ and $\mathcal{X}_2 := \mathcal{X} \setminus \mathcal{X}_1$. One can see that \mathcal{X}_i 's are law invariant and conic; consequently, therefore is $\mathcal{A} := (\mathcal{X}_1 \cap \mathcal{A}_\alpha^0) \cup (\mathcal{X}_2 \cap \mathcal{A}_\alpha^-)$. First, for any random variable $X, \min(F_X^{-1}(z), 0) = F_{-X}^{-1}(z), z \in (0, 1)$ because $F_{\varphi(X)}^{-1}(z) = \varphi(F_X^{-1}(z)), z \in (0, 1)$ for any continuous and increasing function φ (see endnote 4). Second, it is obvious that $\mathcal{A}_\alpha^- \subseteq \mathcal{A} \subseteq \mathcal{A}_\alpha^0$. Third, we will show that \mathcal{A} is surplus invariant. Consider any $X_1 \in \mathcal{A}$ and $X_2 \in \mathcal{X}$ such that $X_2^- \leq X_1^-$ a.s. There are two cases: (1) If $X_1 \in \mathcal{X}_1$, then $X_1 \in \mathcal{A}_\alpha^0$. Because $X_2^- \leq X_1^-$ a.s. and \mathcal{A}_α^0 is surplus invariant, we conclude that $X_2 \in \mathcal{A}_\alpha^0$. Furthermore, $\min(F_{X_1}^{-1}(z), 0) = F_{-X_1}^{-1}(z) \leq F_{-X_2}^{-1}(z) = \min(F_{X_2}^{-1}(z), 0), \forall z \in (0, 1)$ because $X_2^- \leq X_1^-$ a.s. Consequently, $X_2 \in \mathcal{X}_1$ and thus $X_2 \in \mathcal{A}$. (2) If $X_1 \in \mathcal{X}_2$, then $X_1 \in \mathcal{A}_\alpha^-$. Because $X_2^- \leq X_1^-$ a.s. and \mathcal{A}_α^- is surplus invariant, we conclude that $X_2 \in \mathcal{A}_\alpha^- \subseteq \mathcal{A}$. Fourth, as $\mathcal{X} = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, the acceptance set \mathcal{A} is truncation closed. Finally, one can see that $Y \notin \mathcal{A}_\alpha^-$ and $Y \in \mathcal{X}_1 \cap \mathcal{A}_\alpha^0 \subseteq \mathcal{A}$. Moreover, \tilde{Y} with quantile function $F_{\tilde{Y}}^{-1}(z) = z - (1 - \alpha), z \geq 1 - \alpha$, and $F_{\tilde{Y}}^{-1}(z) = -\sqrt{1 - \alpha - z}, z < 1 - \alpha$ falls neither in \mathcal{X}_1 nor in \mathcal{A}_α^- , therefore $\tilde{Y} \notin \mathcal{A}$; however, $\tilde{Y} \in \mathcal{A}_\alpha^0$. To summarize, \mathcal{A} is a surplus-invariant, law-invariant, conic, and truncation-closed acceptance set and $\mathcal{A}_\alpha^- \subseteq \mathcal{A} \subseteq \mathcal{A}_\alpha^0$.

Endnotes

¹ Koch-Medina et al. (2015) use a slightly different, but equivalent, definition of surplus invariance; see Equation (1.1) in their paper and Proposition 1 in Koch-Medina et al. (2017).

² A risk measure ρ is excess invariant if $\rho(X) = \rho(Y)$ for all X and Y such that $X^- = Y^-$ and is loss dependent if $\rho(X) = \rho(\min(X, 0))$ for any X .

³Koch-Medina et al. (2017, Proposition 5) show that a closed acceptance set \mathcal{A} is numéraire invariant and monotone (i.e., for any X and Y , if $X \leq Y$ a.s. and $X \in \mathcal{A}$, then $Y \in \mathcal{A}$) if and only if it is surplus invariant and conic.

⁴In fact, by Lemma A.27 in Föllmer and Schied (2016), $F_{\varphi(X)}^{-1}(z) = \varphi(F_X^{-1}(z))$ for a.e. $z \in (0, 1)$. Since $F_{\varphi(X)}^{-1}(z)$ and $\varphi(F_X^{-1}(z))$ are left continuous, it follows that $F_{\varphi(X)}^{-1}(z) = \varphi(F_X^{-1}(z))$ for any $z \in (0, 1)$. Similar argument leads to $F_{\psi(X)}^{-1}(z) = \psi(F_X^{-1}((1-z)^+))$, $\forall z \in (0, 1)$.

⁵This example also illustrates that for a surplus-invariant acceptance set, conicity is not equivalent to numéraire invariance without any topological assumption on the set. For a comparison, see Koch-Medina et al. (2017, Proposition 5).

⁶We are grateful to an anonymous referee for phrasing most of the sentences of this paragraph in his or her referee reports for our paper.

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