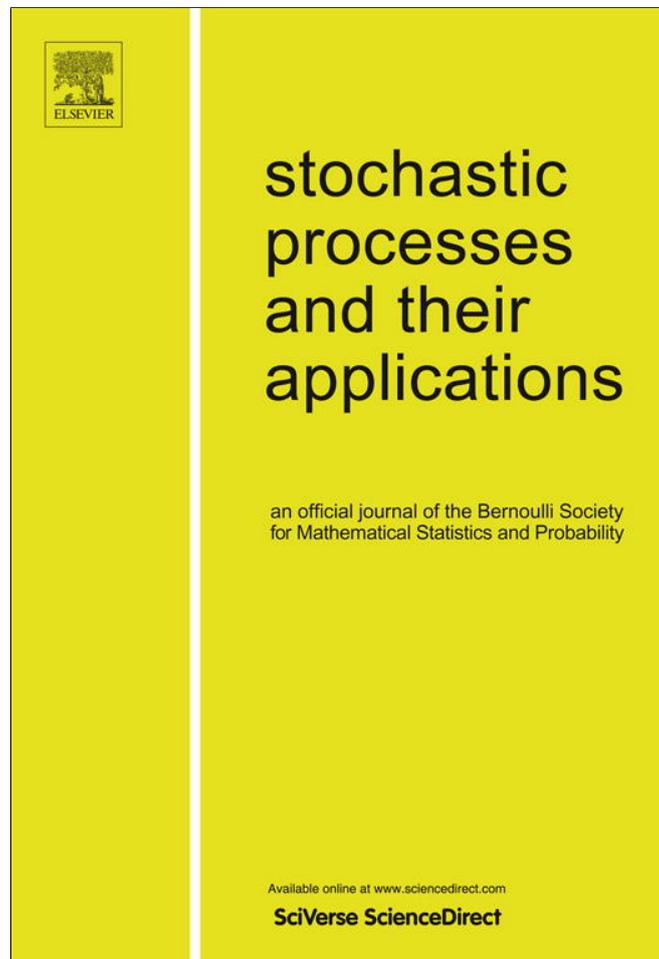


Provided for non-commercial research and education use.
Not for reproduction, distribution or commercial use.



(This is a sample cover image for this issue. The actual cover is not yet available at this time.)

This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:

<http://www.elsevier.com/copyright>



On the Wiener–Hopf factorization for Lévy processes with bounded positive jumps

A. Kuznetsov^{a,*}, X. Peng^b

^a *Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario, M3J 1P3, Canada*

^b *Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong*

Received 23 September 2011; received in revised form 25 April 2012; accepted 26 April 2012
Available online 5 May 2012

Abstract

We study the Wiener–Hopf factorization for Lévy processes with bounded positive jumps and arbitrary negative jumps. We prove that the positive Wiener–Hopf factor can be expressed as an infinite product involving solutions to the equation $\psi(z) = q$, where ψ is the Laplace exponent. Under additional regularity assumptions on the Lévy measure we obtain an asymptotic expression for these solutions. When the process is spectrally negative with bounded jumps, we derive a series representation for the scale function. In order to illustrate possible applications, we discuss the implementation of numerical algorithms and present the results of several numerical experiments.

© 2012 Elsevier B.V. All rights reserved.

MSC: 60G51

Keywords: Lévy process; Wiener–Hopf factorization; Entire functions of Cartwright class; Distribution of the supremum; Spectrally-negative processes; Scale function

1. Introduction

Assume that we want to study the way in which one-dimensional Lévy process X exits a half-line or a finite interval. For example, we might be interested in the first passage time across

* Corresponding author.

E-mail addresses: kuznetsov@mathstat.yorku.ca (A. Kuznetsov), maxhpeng@ust.hk (X. Peng).

a barrier, the overshoot/undershoot at the first passage time, the last time that the process was closest to the barrier, the location of the process at this time, etc. These questions are usually referred to as “exit problems” in the literature, and they have stimulated a lot of research in recent years due to numerous applications in such diverse areas as actuarial mathematics, mathematical finance, queueing theory and optimal control.

Let us denote the supremum $S_t := \sup\{X_s : 0 \leq s \leq t\}$ and infimum $I_t := \inf\{X_s : 0 \leq s \leq t\}$, and let $e(q)$ be an exponentially distributed random variable with parameter $q > 0$, which is independent of the process X . It is an established fact that exit problems are closely related to the Wiener–Hopf factorization, which studies the distribution of $S_{e(q)}$ and $I_{e(q)}$. For example, if we know the positive Wiener–Hopf factor (which is defined as the characteristic function of $S_{e(q)}$), then through the Pecherskii–Rogozin identity [31] we know the joint Laplace transform of the first passage time and the overshoot. The bad news is that for general Lévy processes the Wiener–Hopf factors cannot be obtained in closed form, therefore the best that we can do is to try to find rich enough families of Lévy processes with special analytical properties, for which we can say something useful about the distribution of $S_{e(q)}$ and $I_{e(q)}$.

Let us look at the existing examples of Lévy processes for which one can identify the Wiener–Hopf factors and the distribution of extrema. In the general case, when the process has jumps of both sides, this list includes processes with jumps of rational transform [29,32] and the meromorphic processes, which were introduced recently in [21]. The first class includes processes with hyper-exponential [7,14,17] and phase-type jumps [1,2], while meromorphic processes include Lamperti-stable processes [4,5,8,30], hypergeometric processes [6,22,26], β -processes [18] and θ -processes [19]. In the simpler case when the process is spectrally negative (which means essentially that it has only negative jumps) it turns out that both of the same two classes provide analytically tractable formulas, however in this case there also exist other interesting families, such as the processes constructed in [13] (see also [23]).

One may wonder why these particular processes are so special that it is possible to find the Wiener–Hopf factorization explicitly. It turns out that in all cases the Laplace exponent, defined as $\psi(z) := \ln(\mathbb{E}[\exp(zX_1)])$, has some analytical structure which allows to factorize it as a product of two functions, which are analytic in the left/right complex halfplane. It is not surprising that the analytic structure of $\psi(z)$ plays such an important role, as there is a close connection between the Wiener–Hopf factorization and the Riemann boundary value problems, see [12,20] and the references therein. For example, if the process has hyper-exponential jumps [7], then $\psi(z)$ is a rational function; and if X is a meromorphic process, then $\psi(z)$ is a meromorphic function of a very special type; in both cases these functions can be easily factorized as products of two functions. One can formulate a general “meta-theorem”: Wiener–Hopf factorization can be obtained explicitly if and only if $\psi(z)$ can be extended to a meromorphic function in the left or right complex halfplane. This principle helps to explain why no one has yet produced an explicit Wiener–Hopf factorization for one of the processes which are widely used in mathematical finance, such as VG, CGMY/KoBoL or generalized tempered stable processes (see [9] and the references therein for more information about these families of Lévy processes). It turns out that in all these cases the Laplace exponent has a logarithmic or algebraic branch point in the complex plane, and, therefore, cannot be extended meromorphically. At the same time, we can use this meta-theorem to produce a large class of processes for which there is some hope to have an explicit Wiener–Hopf factorization: if the process has bounded jumps then it follows quite easily from the Lévy–Khinchine formula that the Laplace exponent $\psi(z)$ is an entire function, and it might be possible to factorize it as a product of two functions and obtain some useful information about the Wiener–Hopf factorization.

In this paper we consider a more general class: Lévy processes with bounded positive jumps. There are two main reasons, one theoretical and one more practical, why we are interested in studying this class of processes. First of all, one can see that this is a very large class. In a certain sense it is “dense” in the class of all Lévy processes: clearly, the Lévy measure of any Lévy process X can be truncated at a large positive number, which will result in a process \tilde{X} with bounded positive jumps, which is “close” to X . Therefore studying the Wiener–Hopf factorization for this class may lead to a better understanding of related results for general Lévy processes. The second reason is that there are several situations where processes with bounded positive or negative jumps would be natural candidates for modeling purposes. One important example is the ruin problem for the insurance company which is protected by the reinsurance agreement. In this case the size of each claim is essentially capped at a fixed level, and the amount of the claim above this level is being covered by the reinsurer. The value of the insurance company can be conveniently modeled by a spectrally negative Lévy process with bounded jumps, and now we have an interesting problem of how to compute numerically such important quantities as the ruin probability, discounted penalty function, etc.

It is instructive to draw a parallel with the results of Lewis and Mordecki [29] on processes with positive jumps of rational transform (see also the recent paper by Fourati [12] on the double-sided exit problem for this class of processes). In their case the Laplace exponent of the ascending ladder process $\kappa(q, z)$ (see [25] for the definition of this object) is a rational function, with all singularities lying in the left half-plane $\operatorname{Re}(z) < 0$. In our case it will turn out that $\kappa(q, z)$ is an entire function of a very special type: it belongs to the so-called Cartwright class (see [27,28] and Section 5). This makes it possible to factor this function as an infinite product and to identify the Wiener–Hopf factors. There are also some similarities between the analytical structure for Lévy processes with bounded positive jumps and meromorphic processes [21]. In both cases the positive Wiener–Hopf factor is given as an infinite product involving the solutions to the equation $\psi(z) = q$ in the half-plane $\operatorname{Re}(z) > 0$. The major difference is that in the case of meromorphic processes the solutions to the equation $\psi(z) = q$ are all real and simple, while they are complex when the process has bounded positive jumps, and this fact makes the analytical theory more interesting and the computations somewhat more challenging.

The paper is organized as follows: in Section 2 we present our main results on the Wiener–Hopf factorization for processes with bounded positive jumps, and we obtain an expression for the Wiener–Hopf factors as an infinite product in terms of the solutions to $\psi(z) = q$. We also study the asymptotics of these solutions, which will turn out to be very important for applications and numerical computations. In Section 3 we consider the spectrally negative case, and we obtain a series representation for the scale function $W^{(q)}(x)$. A brief discussion of the numerical methods and the results of several numerical experiments are presented in Section 4, while Section 5 contains the proofs of all results.

2. Processes with bounded positive jumps

Let us first introduce some notations and definitions. The Lévy measure of the process X will be denoted by $\Pi(dx)$, and we will use the following notations for its tails: $\bar{\Pi}^+(x) := \Pi((x, \infty))$ for $x > 0$ and $\bar{\Pi}^-(x) := \Pi((-\infty, x))$ for $x < 0$. In this paper we consider the class of processes with bounded positive jumps, thus we will assume that the Lévy measure Π has support on $(-\infty, k]$. Here k is the right boundary of the support of Π , that is

$$k := \inf\{x > 0 : \bar{\Pi}^+(x) = 0\}. \quad (1)$$

We will also assume that $k > 0$, so that we exclude the spectrally negative case, which will be considered in the next section. Note that at this stage we do not impose any restrictions on the Lévy measure on the negative half-line.

The Laplace exponent of the process X is defined as $\psi(iz) := \ln(\mathbb{E}[\exp(izX_1)])$ for $z \in \mathbb{R}$, and it can be expressed via the Lévy–Khintchine formula as follows

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + \int_{-\infty}^k (e^{zx} - 1 - zxh(x)) \Pi(dx), \tag{2}$$

where $\sigma \geq 0$, $\mu \in \mathbb{R}$ and $h(x)$ is the cutoff function. When the process has jumps of bounded variation, or equivalently, when

$$\int_{-1}^1 |x| \Pi(dx) < \infty, \tag{3}$$

we will assume that $h(x) \equiv 0$, then μ can be interpreted as the linear drift of the process. When the jump part of the process has infinite variation, or equivalently, when condition (3) is violated, we will assume that $h(x) \equiv \mathbf{1}_{\{x > -1\}}$ (or $h(x) \equiv 1$ if $\mathbb{E}[|X_1|]$ is finite). Note that formula (2) implies that the Laplace exponent $\psi(z)$ can be analytically continued into the half-plane $\text{Re}(z) > 0$. Also note that $\psi(z)$ is real when $z > 0$, and that $\overline{\psi(z)} = \psi(\bar{z})$. In particular, the last property implies that if for some $q \in \mathbb{R}$ the number $z \in \mathbb{C}$ is a solution to the equation $\psi(z) = q$, then so is \bar{z} .

Everywhere in this paper we will denote the first quadrant of the complex plane as

$$\mathcal{Q}_1 := \{z \in \mathbb{C} : \text{Re}(z) > 0, \text{Im}(z) > 0\},$$

and we will always use the principal branch of the logarithm and the power function, that is the branch cut will be taken along the negative half-line and for all $z \in \mathbb{C}$ we have $\arg(z) \in (-\pi, \pi]$.

2.1. Analytic properties of the Wiener–Hopf factors

The following theorem is our first main result. It describes the analytic structure of the Wiener–Hopf factors for processes with bounded positive jumps.

Theorem 1. *Assume that $q > 0$. Equation $\psi(z) = q$ has a unique positive solution ζ_0 and infinitely many solutions in \mathcal{Q}_1 , which we denote by $\{\zeta_n\}_{n \geq 1}$. Assume that ζ_n are arranged in the order of increase of the absolute value. The following statements are true:*

- (i) ζ_0 has multiplicity one and $\text{Re}(\zeta_n) \geq \zeta_0$ for all $n \geq 1$.
- (ii) The series $\sum_{n \geq 1} \text{Re}(\zeta_n^{-1})$ converges.
- (iii) All of the numbers $\{\zeta_n\}_{n \geq 1}$, except possibly those of a set of zero density, lie inside arbitrary small angle $\pi/2 - \epsilon < \arg(z) < \pi/2$, and the density of zeros inside this angle is equal to

$$\lim_{r \rightarrow +\infty} \frac{\#\{\zeta_n : |\zeta_n| < r \text{ and } \pi/2 - \epsilon < \arg(\zeta_n) < \pi/2\}}{r} = \frac{k}{2\pi}. \tag{4}$$

- (iv) The Wiener–Hopf factors can be identified as follows: for $\text{Re}(z) \geq 0$

$$\begin{cases} \phi_q^+(iz) := \mathbb{E} \left[e^{-zS_{e(q)}} \right] = e^{\frac{kz}{2}} \left(1 + \frac{z}{\zeta_0} \right)^{-1} \prod_{n \geq 1} \left(1 + \frac{z}{\zeta_n} \right)^{-1} \left(1 + \frac{z}{\bar{\zeta}_n} \right)^{-1}, \\ \phi_q^-(-iz) := \mathbb{E} \left[e^{zI_{e(q)}} \right] = \frac{q}{q - \psi(z)} \frac{1}{\phi_q^+(-iz)}. \end{cases} \tag{5}$$

The proof of **Theorem 1** can be found in Section 5.

Let us present a very simple example, which will illustrate the results presented in **Theorem 1**. Consider a process $X_t = kN_t$, where N_t is the standard Poisson process. It is clear that X is a process with bounded positive jumps, and that its Laplace exponent is $\psi(z) = \exp(kz) - 1$. Solving equation $\psi(z) = q$ for $q > 0$ we find that

$$\zeta_0 = \frac{\ln(1 + q)}{k}, \quad \zeta_n = \frac{\ln(1 + q)}{k} + \frac{2n\pi i}{k}, \quad n \geq 1.$$

Since $\text{Re}(\zeta_n^{-1}) = O(n^{-2})$, the series $\sum_{n \geq 1} \text{Re}(\zeta_n^{-1})$ converges, thus we have checked part (ii) of the **Theorem 1**. Next, all the zeros belong to the vertical line $\text{Re}(z) = \zeta_0$, they are equidistant and the spacing between them is equal to $2\pi/k$. This confirms statement (iii): all zeros (except for a finite number) lie inside arbitrary small angle $\pi/2 - \epsilon < \arg(z) < \pi/2$, and the density of zeros inside this angle, which is inversely proportional to the spacing, is equal to $k/(2\pi)$. Finally, since X is a subordinator, we have $S_{e(q)} \equiv X_{e(q)}$, thus

$$\begin{aligned} \mathbb{E} \left[e^{-zS_{e(q)}} \right] &= \mathbb{E} \left[e^{-zX_{e(q)}} \right] = \frac{q}{q - \psi(-z)} \\ &= \frac{q}{1 + q - e^{-kz}} = e^{\frac{kz}{2}} \frac{\sinh\left(\frac{1}{2} \ln(1 + q)\right)}{\sinh\left(\frac{1}{2}(kz + \ln(1 + q))\right)}, \end{aligned}$$

and we see that the infinite product representation in (5) is equivalent to the well-known infinite product formula for the hyperbolic sine function.

It is also easy to verify the validity of **Theorem 1** for a more general class of processes with double-sided jumps. Let us assume that for some $h > 0$ the measure $\Pi(dx)$ is supported on a finite subset of a lattice $h\mathbb{Z}$, that is there exist $m, l \in \mathbb{N}$ such that the support of $\Pi(dx)$ is equal to $\{-mh, -(m - 1)h, \dots, -h, h, \dots, (l - 1)h, lh\}$. In this case the right boundary of the support of the Lévy measure is $k = lh$. Let X be a compound Poisson process defined by the measure $\Pi(dx)$ (note that X can be constructed as a linear combination of $m + l$ independent Poisson processes). From the Lévy–Khintchine formula (2) we find that the Laplace exponent of X is given by

$$\psi(z) = \sum_{j=1}^m \Pi(\{-jh\}) \left(e^{-jh z} - 1 \right) + \sum_{j=1}^l \Pi(\{jh\}) \left(e^{jh z} - 1 \right).$$

Note that the function $\psi(\ln(w)/h)$ is a rational function, therefore using the change of variable $z = \ln(w)/h$ the equation $\psi(z) = q$ can be transformed into a polynomial equation of degree $m + l$. It is possible to prove (we leave it as an exercise) that this polynomial equation has exactly m solutions inside the open unit circle $\{w \in \mathbb{C} : |w| < 1\}$ and exactly l solutions $\{w_1, \dots, w_l\}$ outside the closed unit circle. The solutions ζ_n to the original equation $\psi(z) = q$ can now be found by solving the equation $\exp(hz) = w_j$, thus they are given by

$$\left\{ \frac{\ln(w_j)}{h} + \frac{2n\pi i}{h}; n \in \mathbb{Z}, 1 \leq j \leq l \right\}.$$

Again, it is easy to check that the series $\sum_{n \geq 1} \text{Re}(\zeta_n^{-1})$ converges. Also, the solutions lie on l vertical lines, therefore all of them (except for a finite number) lie inside arbitrary small angle

$\pi/2 - \epsilon < \arg(z) < \pi/2$, and the density of zeros inside this angle is equal to $l \times h/(2\pi) = k/(2\pi)$. Infinite product representation for the positive Wiener–Hopf factor (5) is again equivalent to elementary infinite product expressions for certain trigonometric functions.

In the above two examples we were able to describe the solutions to the equation $\psi(z) = q$ in a very precise form, but in the general case this will be a complicated transcendental equation and there is little hope to obtain as much information about ζ_n . However, as our next result shows, we can obtain some very useful information about the asymptotic behavior of ζ_n , provided that the Laplace exponent $\psi(z)$ has regular growth as $z \rightarrow \infty$. The connection between the regularity of growth and the regularity of distribution of zeros of an entire function is well-known, see, for example, Chapters 2 and 3 in [27]. The main idea is that analytic functions which grow regularly at infinity also enjoy certain regularity in the distribution of zeros (and, in fact, the opposite is also true). In the following theorem we impose a rather strong regularity condition on the growth of the Laplace exponent in the half-plane $\operatorname{Re}(z) > 0$ in order to obtain an explicit asymptotic approximation for the solutions to $\psi(z) = q$. This asymptotic expression for ζ_n would prove to be very useful in the next section, when we will derive a series representation for the scale function $W^{(q)}(x)$, and later in Section 4, when we will discuss numerical algorithms.

Theorem 2. *Assume that*

$$\psi(z) = Ae^{kz}z^{-a} + Bz^b + o\left(e^{kz}z^{-a}\right) + o\left(z^b\right), \tag{6}$$

as $z \rightarrow \infty$ in the domain \mathcal{Q}_1 . Let us also assume that $a \geq 0$ and $b > 0$. Then all sufficiently large solutions to $\psi(z) = q$ are simple and there exists $m \in \mathbb{Z}$ such that

$$\begin{aligned} \zeta_{n+m} &= \frac{1}{k} \left[\ln \left(\left| \frac{B}{A} \right| \right) + (a+b) \ln \left(\frac{2n\pi}{k} \right) \right] \\ &\quad + \frac{i}{k} \left[\arg \left(\frac{B}{A} \right) + \left(\frac{1}{2}(a+b) + 2n + 1 \right) \pi \right] + o(1) \end{aligned} \tag{7}$$

as $n \rightarrow +\infty$.

The proof of Theorem 2 is presented in Section 5.

Remark 1. Note that formula (7) implies that $\zeta_n = (a+b) \ln(n)/k + 2\pi ni/k + O(1)$ as $n \rightarrow \infty$, which again confirms statements (ii) and (iii) of Theorem 1: the zeros cluster “close” to the imaginary axis, or to say it more precisely, $\arg(\zeta_n) \nearrow \pi/2$ as $n \rightarrow \infty$; and secondly, the density of zeros in \mathcal{Q}_1 (which is inversely proportional to the average spacing between them) is equal to $k/(2\pi)$.

Condition $b > 0$ in Theorem 2 implies that $\psi(iz) \rightarrow \infty$ as $z \rightarrow \infty$, $z \in \mathbb{R}$, therefore X cannot be a compound Poisson process (see Proposition 2, page 16, in [3]). This shows that the two examples considered on page 4 do not satisfy the conditions of Theorem 2, however, if we take these compound Poisson processes and add a drift (or a Brownian motion with drift), then it is easy to check that the Laplace exponent of this perturbed process will satisfy (6). A natural problem then is to find sufficient conditions on the triple $\{\mu, \sigma, H\}$, which will ensure that $\psi(z)$ satisfies the asymptotic relation (6). Below we present a set of sufficient conditions.

Definition 1. We will say that a real function $f(x)$ is piecewise n -times continuously differentiable on an interval $[a, b]$ if there exists a finite set of numbers $\{x_k\}_{1 \leq k \leq m}$, such that

- (i) $a = x_1 < x_2 < \dots < x_m = b$,

- (ii) $f \in \mathcal{C}^n([a, b] \setminus \{x_1, x_2, \dots, x_m\})$,
- (iii) for each $j = 0, 1, \dots, n$ and $k = 1, 2, \dots, m$ there exist left and right limits $f^{(j)}(x_k-)$ and $f^{(j)}(x_k+)$.

We will use the notation $f \in \mathcal{PC}^n[a, b]$. In the case of an open interval (a, b) the definition of $\mathcal{PC}^n(a, b)$ is very similar, except for condition $a < x_1$ and $x_m < b$. Similarly, one can define the remaining cases of intervals $(a, b]$ and $[a, b)$.

Definition 2. We say that a Lévy measure is *regular* if the following two conditions are satisfied:

- (1) There exist constants $\hat{C}, \hat{\alpha}$ and $\{\hat{C}_j, \hat{\alpha}_j\}_{1 \leq j \leq \hat{m}}$ such that

$$\bar{\Pi}^-(x) - \hat{C}|x|^{-\hat{\alpha}} - \sum_{j=1}^{\hat{m}} \hat{C}_j|x|^{-\hat{\alpha}_j} = O(1), \quad x \rightarrow 0^-, \tag{8}$$

where $\hat{\alpha}, \hat{\alpha}_j \in (-\infty, 2) \setminus \{0, 1\}$ and $\hat{\alpha}_j < \hat{\alpha}$.

- (2) There exists $n \in \mathbb{N} \cup \{0\}$ such that

- (2a) for some constants C, α and $\{C_j, \alpha_j\}_{1 \leq j \leq m}$ we have

$$\bar{\Pi}^+(x) - Cx^{-\alpha} - \sum_{j=1}^m C_jx^{-\alpha_j} \in \mathcal{PC}^{n+1}[0, k], \tag{9}$$

where $\alpha, \alpha_j \in (-\infty, 2) \setminus \{0, 1\}$ and $\alpha_j < \alpha$;

- (2b) $\bar{\Pi}^{+(n)}(k-) \neq 0$;

- (2c) $\bar{\Pi}^+(x) \in \mathcal{C}^{n-1}(\mathbb{R}^+)$ (this condition is not needed for $n = 0$).

Remark 2. Note that conditions (1) and (2a) imply that the Blumenthal–Gettoor index

$$\beta(\Pi) := \inf \left\{ \gamma > 0 : \int_{-1}^1 |x|^\gamma \Pi(dx) < \infty \right\} \tag{10}$$

is equal to $\beta(\Pi) = \max(\alpha, \hat{\alpha}, 0)$.

Definition 2 is not very easy to interpret, therefore we will try to give some intuition behind these conditions. Conditions (1) and (2a) guarantee that the Lévy measure is sufficiently well-behaved in the neighborhood of zero. This will ensure that the leading term in the asymptotics of $\psi(z)$ as $z \rightarrow \infty$ grows exactly as z^b , and that it does not contain any logarithmic terms. Conditions (2b) and (2c) are slightly harder to interpret. Essentially, they mean that the Lévy measure restricted to \mathbb{R}^+ has its “worst” possible singularity at the right-end point of its support. In order to illustrate these conditions and explain why they are needed, let us consider the following simple example.

Example 1. Assume that the Lévy measure is given by

$$\Pi(dx) = \mathbf{1}_{\{x < 0\}}e^x dx + \mathbf{1}_{\{0 < x < 4\}}dx + \delta_3(dx), \tag{11}$$

so that $\Pi(dx)$ has an atom of mass one at $x = 3$. Because of the atom at $x = 3$ we know that $\bar{\Pi}^+(x)$ is not continuous, therefore we are forced to take $n = 0$ in Definition 2. But since we have no atom at $x = k = 4$, we find that $\bar{\Pi}^+(k-) = 0$, which violates condition (2b), thus we conclude that the measure $\Pi(dx)$ is not regular.

Next, let us check that if X is a Lévy process which has the Lévy measure (11), linear drift $\mu = 1$ and no Gaussian component, then the Laplace exponent of X does not satisfy (6).

We compute the Laplace exponent using the Lévy–Khintchine formula (2) and find that it has asymptotics

$$\psi(z) = z + \frac{e^{4z}}{z} + e^{3z} + O(1),$$

as $z \rightarrow \infty, \operatorname{Re}(z) > 0$. Now, it is easy to see that in the domain $0 < \operatorname{Re}(z) < \frac{1}{2} \ln |z|$ we have $\psi(z) = z + e^{3z} + o(e^{3z})$, while in the domain $\operatorname{Re}(z) > 2 \ln |z|$ we have $\psi(z) = e^{4z}/z + o(e^{4z}/z)$. This implies that we cannot find a single uniform asymptotic formula for $\psi(z)$, which would be valid in the entire first quadrant \mathcal{Q}_1 , as in (6). One can see that happens because the “worst” singularity of $\Pi(dx)$, which is the atom at $x = 3$, is not located at the right boundary $k = 4$.

One can also check that the asymptotic expression (6) will be satisfied if we replace $\delta_3(dx)$ by $\delta_4(dx)$ in (11) (or if we just add a second atom at $x = 4$), and at the same time this will also give us a regular Lévy measure according to Definition 2.

In the following example we exhibit a large family of regular Lévy measures (it is an easy exercise to verify all the conditions of Definition 2).

Example 2. Assume that the Lévy measure $\Pi(dx)$ has a density $\pi(x)$ given by

$$\pi(x) := \mathbf{1}_{\{x < 0\}} \hat{f}(x) |x|^{-1-\hat{\alpha}} + \mathbf{1}_{\{0 < x < k\}} f(x) x^{-1-\alpha}, \tag{12}$$

where $\alpha, \hat{\alpha} \in (-\infty, 2) \setminus \{0, 1\}$ and the functions f, \hat{f} satisfy the following conditions: (i) $f(x)$ and $\hat{f}(x)$ can be represented by convergent Taylor series in some neighborhood of zero; (ii) $f(x)$ is $\mathcal{P}C^1[0, k]$; (iii) $f(k-) > 0$. Then $\Pi(dx)$ is regular.

The above example shows that there are indeed many interesting Lévy processes with regular Lévy measures. For example, we can take one of the processes, which are widely used in Mathematical Finance (such as CGMY/KoBoL or generalized tempered stable, see [9]), truncate its Lévy measure at any positive number and this will give us a regular Lévy measure.

The next proposition shows that if the Lévy process has a regular Lévy measure, then its Laplace exponent satisfies the asymptotic expansion (6), and therefore the roots ζ_n have simple asymptotic approximation given by (7).

Proposition 1. Assume that X is not a compound Poisson process and that the Lévy measure of X is regular. Then the asymptotic expression (6) is true, with parameters $A = (-1)^n \bar{\Pi}^{+(n)}(k-)$ and $a = n$. The parameters B and b can be identified as follows:

- (i) if $\sigma > 0$, then $B = \sigma^2/2$ and $b = 2$,
- (ii) if the process has paths of bounded variation and $\mu \neq 0$, then $B = \mu$ and $b = 1$.

In the remaining cases, when the process has paths of unbounded variation and $\sigma = 0$, or when the process has paths of bounded variation and $\mu = 0$, we have $b = \beta(\Pi) = \max(\alpha, \hat{\alpha})$ and

$$B = \begin{cases} -C e^{-\pi i \alpha} \Gamma(1 - \alpha), & \text{if } \alpha > \hat{\alpha}, \\ -\hat{C} \Gamma(1 - \hat{\alpha}), & \text{if } \alpha < \hat{\alpha}, \\ -(C e^{-\pi i \alpha} + \hat{C}) \Gamma(1 - \alpha), & \text{if } \alpha = \hat{\alpha}. \end{cases}$$

Moreover, the asymptotic expression for $\psi'(z)$ can be obtained from (6) by taking derivative of the right-hand side.

The proof of Proposition 1 can be found in Section 5.

2.2. Partial fraction decomposition and distribution of $S_{e(q)}$

As we have mentioned in Section 1, the positive Wiener–Hopf factor is a rational function in the case when X has positive jumps of rational transform (see [29]). By performing the partial fraction decomposition of this rational function and inverting the Laplace transform one can easily obtain the distribution of $S_{e(q)}$ in closed form. The same procedure works for meromorphic processes (see [21]), though in this case we must work with meromorphic functions instead of rational functions, and things become slightly more technical. In our case, when the process has bounded positive jumps, formula (5) tells us that the positive Wiener–Hopf factor $\phi_q^+(iz)$ is also a meromorphic function, thus we might hope to follow the same procedure and obtain a series representation for the distribution of $S_{e(q)}$, which would be very useful for applications. Unfortunately, it turns out that proving existence of the partial fraction decomposition for $\phi_q^+(iz)$ of the form (5) is a much harder problem, and we were not able to give a completely rigorous proof of such a result or to find such a result in the existing literature. However, if one assumes that such a partial fraction decomposition exists, it is rather easy to obtain the form of its coefficients and the resulting expression for the distribution of $S_{e(q)}$. Let us sketch here the main steps, and later, in Section 4 we perform several numerical experiments which seem to confirm our conjecture.

First of all, we note that it is very likely that all the zeros $\{\zeta_n\}_{n \geq 1}$ of the function $\psi(z) - q$ are simple. This was proved in Theorem 2 for ζ_0 and for large ζ_n . For other roots one could use the following (rather informal) argument. We know that z is a solution of $\psi(z) = q$ with multiplicity greater than one if and only if $\psi'(z) = 0$. We can rephrase this statement: equation $\psi(z) = q$ has a solution of multiplicity greater than one if only if $q = \psi(\zeta)$, where ζ is a root of $\psi'(z)$. But $\psi'(z)$ is analytic in the half-plane $\text{Re}(z) > 0$, thus it has a discrete set of zeros, and for any given complex root of $\psi'(z)$ there is no reason to expect that $\psi(\zeta)$ will be a real positive number, thus it seems very unlikely that there should exist $q > 0$ such that $\psi(z) - q$ has a multiple root. Even if such a value of q exists, we see that in the worst possible case, there can be only finitely many such values. This shows that the assumption that all the zeros $\{\zeta_n\}_{n \geq 1}$ are simple should not be very restrictive for practical purposes. However, of course this fact would require a rigorous proof in the future.

Next, assuming that 0 is regular for $(0, \infty)$, or equivalently, that the distribution of $S_{e(q)}$ has no atom at zero, we must have $\phi_q^+(iz) = \mathbb{E}[\exp(-zS_{e(q)})] \rightarrow 0$ as $\text{Re}(z) \rightarrow +\infty$, thus it is reasonable to expect that the partial fraction decomposition for $\phi_q^+(iz)$ should be of the form

$$\begin{aligned} \phi_q^+(iz) &= \mathbb{E}\left[e^{-zS_{e(q)}}\right] = e^{\frac{kz}{2}} \left(1 + \frac{z}{\zeta_0}\right)^{-1} \prod_{n \geq 1} \left(1 + \frac{z}{\zeta_n}\right)^{-1} \left(1 + \frac{z}{\bar{\zeta}_n}\right)^{-1} \\ &= \frac{a_0}{z + \zeta_0} + \sum_{n \geq 1} \left[\frac{a_n}{z + \zeta_n} + \frac{\bar{a}_n}{z + \bar{\zeta}_n} \right] \end{aligned} \tag{13}$$

where $a_n = \text{Res}(\phi_q^+(iz) : z = -\zeta_n)$ for $n \geq 0$. Using the above infinite product representation we can easily compute the residues at points ζ_n and obtain

$$a_0 = \zeta_0 e^{-\frac{1}{2}k\zeta_0} \prod_{m \geq 1} \left| 1 - \frac{\zeta_0}{\zeta_m} \right|^{-2},$$

and

$$a_n = \frac{\zeta_0 |\zeta_n|^2 e^{-\frac{1}{2}k\zeta_n}}{2\text{Im}(\zeta_n)(\zeta_n - \zeta_0)} \prod_{\substack{m \geq 1 \\ m \neq n}} \left[\left(1 - \frac{\zeta_n}{\zeta_m}\right) \left(1 - \frac{\zeta_n}{\bar{\zeta}_m}\right) \right]^{-1}.$$

Therefore, provided that there exists a partial fraction decomposition of the form (13), we can use the uniqueness of Laplace transform and conclude that the density of the supremum $S_{e(q)}$ should be given by the following infinite series

$$\frac{d}{dx} \mathbb{P}(S_{e(q)} \leq x) = a_0 e^{-\zeta_0 x} + 2 \sum_{n \geq 1} \text{Re} [a_n e^{-\zeta_n x}]. \tag{14}$$

Again, we emphasize that at this point the existence of the partial fraction decomposition (13) is just a conjecture, which would have to be established rigorously in the future work.

3. Scale functions for spectrally negative processes with bounded jumps

Everywhere in this section we will assume that Y is a spectrally negative Lévy process with bounded jumps, so that the Lévy measure $\Pi_Y(dx)$ is supported on the interval $[-k, 0)$ where $k > 0$, and $-k$ is the left boundary of the support of $\Pi_Y(dx)$. From the Lévy–Khintchine formula (2) it follows that in this case the Laplace exponent $\psi_Y(z) = \ln \mathbb{E}[\exp(zY_1)]$ is given by

$$\psi_Y(z) = \frac{1}{2} \sigma^2 z^2 + \mu z + \int_{-k}^0 (e^{zx} - 1 - zxh(x)) \Pi_Y(dx),$$

and we see that $\psi_Y(z)$ is an entire function, which is convex for real values of z . Since $\psi_Y(0) = 0$, it is clear that for each $q > 0$ the equation $\psi_Y(z) = q$ has a unique positive solution $z = \Phi(q)$, and in fact it is known from the general theory of spectrally negative processes that this is a unique solution in the half-plane $\text{Re}(z) > 0$, see Chapter 8 in [25].

For $q > 0$ the scale function $W^{(q)}(x)$ is defined as follows: $W^{(q)}(x) = 0$ for $x < 0$ and on $[0, \infty)$ it is characterized via the Laplace transform identity

$$\int_0^\infty e^{-zx} W^{(q)}(x) dx = \frac{1}{\psi_Y(z) - q}, \quad \text{Re}(z) > \Phi(q). \tag{15}$$

The scale function can be considered as the main building block for the multitude of fluctuation identities for spectrally negative processes, see [25,23] for many examples of such identities. Here we will present one fundamental identity, which is related to the exit of the process Y from an interval. If we define the first passage time $\tau_a^+ := \inf\{t > 0 : Y_t > a\}$ and similarly $\tau_0^- := \inf\{t > 0 : Y_t < 0\}$, then Theorem 8.1 in [25] tells us that

$$\mathbb{E}_x \left[e^{-q\tau_a^+} \mathbf{1}_{\{\tau_a^+ < \tau_0^-\}} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \quad x \leq a, \quad q \geq 0.$$

In fact, this identity justifies the name “scale function”: we see that $W^{(q)}(x)$ plays an analogous role to scale function for diffusions.

Our main goal in this section is to obtain an expression for the scale function $W^{(q)}(x)$ in terms of the Laplace exponent and the roots of the entire function $\psi_Y(z) - q$. We will consider spectrally negative Lévy processes, whose Lévy measure satisfies the following conditions.

Definition 3. We say that the Lévy measure of a spectrally negative process Y is *regular* if there exists $n \in \mathbb{N} \cup \{0\}$ such that

(a) for some constants C, α and $\{C_j, \alpha_j\}_{1 \leq j \leq m}$ we have

$$\bar{\Pi}^-(x) - C|x|^{-\alpha} - \sum_{j=1}^m C_j|x|^{-\alpha_j} \in \mathcal{PC}^{n+1}[-k, 0],$$

where $\alpha, \alpha_j \in (-\infty, 1) \cup (1, 2)$ and $\alpha_j < \alpha$;

(b) $\bar{\Pi}^{-(n)}((-k)^+) \neq 0$;

(c) $\bar{\Pi}^-(x) \in \mathcal{C}^{n-1}(\mathbb{R}^-)$ (this condition is not needed for $n = 0$).

By considering the dual process $\hat{Y} = -Y$ and using Proposition 1 we see that if Y has a regular Lévy measure then its Laplace exponent satisfies

$$\psi_Y(-z) = Ae^{kz}z^{-a} + Bz^b + o\left(e^{kz}z^{-a}\right) + o\left(z^b\right) \tag{16}$$

as $z \rightarrow \infty$ in the domain \mathcal{Q}_1 , where $a \geq 0$ and $b > 0$. Moreover, the parameters in the asymptotic expression (16) are given by

$$\begin{cases} A = \bar{\Pi}^{-(n)}((-k)^+), & \text{and } a = n, \\ B = \frac{\sigma^2}{2}, & \text{and } b = 2, \text{ if } \sigma > 0, \\ B = -\mu, & \text{and } b = 1, \text{ if } \sigma = 0 \text{ and } \alpha < 1, \\ B = -Ce^{-\pi i \alpha} \Gamma(1 - \alpha), & \text{and } b = \alpha, \text{ if } \sigma = 0 \text{ and } \alpha > 1. \end{cases} \tag{17}$$

Theorems 1 and 2 tell us that equation $\psi_Y(-z) = q$ has infinitely many solutions $\{\zeta_n\}_{n \geq 0}$ in \mathcal{Q}_1 , moreover the first solution ζ_0 is real and positive, we have $\zeta_n \geq \zeta_0$ for $n \geq 1$ and the large solutions ζ_n satisfy asymptotic relation (7) with constants A, a, B and b as given in (17).

The next theorem is our main result in this section. It provides the series representation for the scale function $W^{(q)}(x)$, given in terms of the Laplace exponent $\psi_Y(z)$ and the numbers ζ_n . Its proof can be found in Section 5.

Theorem 3. Assume that (i) $q > 0$ or (ii) $q = 0$ and $\Phi(q) > 0$. If the Lévy measure of Y is regular and all solutions to $\psi_Y(z) = q$ are simple, then for $x > 0$ we have

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{e^{-\zeta_0 x}}{\psi'(-\zeta_0)} + 2 \sum_{n \geq 1} \operatorname{Re} \left[\frac{e^{-\zeta_n x}}{\psi'(-\zeta_n)} \right], \tag{18}$$

where the series converges uniformly on $[\epsilon, \infty)$ for every $\epsilon > 0$.

Formula (18) is in fact very similar to the corresponding expression for the scale function for meromorphic processes, see [24,23].

In the very unlikely case that some of the solutions to $\psi_Y(z) = q$ are not simple (see the above discussion on page 9) the expression on the right-hand side of (18) would have to be modified. The coefficients $\exp(-\zeta_n x)/\psi'_Y(-\zeta_n)$ are the residues of $\exp(zx)/(\psi_Y(z) - q)$ at $z = -\zeta_n$ (see the proof of Theorem 3 in Section 5), and these coefficients would have to be appropriately modified if ζ_n is a root of $\psi_Y(-z) - q$ of multiplicity greater than one.

4. Numerical examples

The main reason why we are interested in the analytical structure of the Wiener–Hopf factorization is that its understanding can lead to efficient numerical algorithms for computing such important objects as the distribution of supremum $S_{e(q)}$ or infimum $I_{e(q)}$, or the scale function $W^{(q)}(x)$. For general Lévy processes, these are very challenging problems. For example, computing the distribution of $S_{e(q)}$ in general case would involve evaluating numerically two integral transforms: first one has to compute the positive Wiener–Hopf factor via the formula (see [29])

$$\mathbb{E} \left[e^{izS_{e(q)}} \right] = \exp \left[\frac{z}{2\pi i} \int_{\mathbb{R}} \ln \left(\frac{q}{q - \psi(iu)} \right) \frac{du}{u(u - z)} \right], \quad \text{Im}(z) > 0,$$

and then to perform the inverse Fourier transform, in order to recover the distribution of $S_{e(q)}$. Similarly, computing the scale function $W^{(q)}(x)$ in general is equivalent to inverting the Laplace transform in (15) (see [23] for the detailed discussion and comparison of several numerical algorithms for computing the scale function).

The results presented in this paper lead to a very different approach for computing the distribution of $S_{e(q)}$ and the scale function $W^{(q)}(x)$. This approach does not rely on the numerical evaluation of multiple integral transforms; instead, the main ingredients are the solutions to the equation $\psi(z) = q$ and infinite series representations (14) and (18). Therefore, this approach is very close in spirit to the techniques that are used for processes with positive jumps of rational transform [29] or meromorphic [21] processes.

Our main goal in this section is to give a brief description of the numerical algorithms and techniques which are suitable for Lévy processes with bounded positive jumps. In particular, we would like to show that series expansions (14) and (18) may lead to efficient numerical computations. However, the detailed investigation of these numerical algorithms is beyond the scope of the current paper and we will leave to future work such important questions as the speed of convergence, rigorous error analysis, etc.

4.1. Preliminaries

In order to implement the numerical algorithms based on formulas (14) and (18), the first step is to solve the following non-trivial problem: how can we find the complex solutions of the equation $\psi(z) = q$? As we will see, this is not an easy problem, yet it can be solved rather efficiently provided that we use the correct techniques.

The main problem in finding the zeros of $\psi(z) - q$ is that all of them (except for ζ_0) are complex numbers. Note that the real zero ζ_0 can be easily found by bisection method, followed by a few steps of Newton’s method. The large zeros of $\psi(z) = q$ satisfy asymptotic relation (7), thus they can be found by Newton’s method which starts from the value on the right-hand side of (7). The only problem that remains is how to compute the complex zeros of $\psi(z) - q$ which are not too large.

The problem of computing the zeros of an analytic function inside a bounded domain has been investigated by many authors, see, for example, [10,34,35] and the references therein. We will follow the method presented in [10], which is based on Cauchy’s Argument Principle. Let us recall this important result. We denote the change in the argument of an analytic function $f(z)$ over a piecewise smooth curve C as

$$\Delta \arg(f, C) := \text{Im} \left[\int_C \frac{f'(z)}{f(z)} dz \right], \tag{19}$$

provided that $f(z) \neq 0$ for $z \in C$. Cauchy's Argument Principle states that if C is a simple closed contour which is oriented counter-clockwise, and $f(z)$ is analytic and non-zero on C and analytic inside C , then $\Delta \arg(f, C) = 2\pi N$, where N is the number of zeros of $f(z)$ inside contour C .

This result leads to a practical iterative procedure which allows to determine the complex zeros of an analytic function $f(z)$ inside a closed contour. We start with an initial rectangle R in the complex plane and proceed through the following sequence of steps: (i) compute $N = N(R)$ —the number of zeros of $f(z)$ inside rectangle R , (ii) if $N = 0$, then we stop, (iii) if $N = 1$ —we try to find the zero using Newton's method started from a point inside the rectangle, (iv) if $N > 1$ or if Newton's method in step (iii) fails—then we subdivide the rectangle R into a finite number of disjoint rectangles R_j . For each of the smaller rectangles R_j we proceed through the same sequence of steps (i) \rightarrow (ii) \rightarrow (iii) \rightarrow (iv). At some point the rectangles which contain the zeros of $f(z)$ will become very small, therefore the starting point of Newton's method will be close to the target and Newton's method would converge to the zero of $f(z)$. This shows that (at least, in theory) we should be able to recover all zeros of $f(z)$ inside R . There are some technical issues which arise when at some step of the algorithm we obtain a rectangle containing a zero which is extremely close to the boundary of this rectangle, however we found that in our numerical experiments this issue was not a big problem. More details on this algorithm and some numerical examples can be found in [10].

Next, let us review some basic facts about the incomplete gamma function, which will be used extensively in this section. The incomplete gamma function is defined for $\text{Re}(s) > 0$ and $z > 0$ as

$$\gamma(s, z) := \int_0^z u^{s-1} e^{-u} du. \tag{20}$$

It is known (see Section 8.35 in [15]) that the function $z \mapsto z^{-s} \gamma(s, z)$ is an entire function and it can be represented by Taylor series (which converge everywhere in the complex plane) in two different ways

$$z^{-s} \gamma(s, z) = s^{-1} {}_1F_1(s, s + 1; -z) = \sum_{n \geq 0} \frac{(-1)^n}{n!} \frac{z^n}{s + n} = e^{-z} \sum_{n \geq 0} \frac{z^n}{(s)_{n+1}}. \tag{21}$$

Here, as usual, $(a)_n := a(a + 1) \cdots (a + n - 1)$ denotes the Pochhammer symbol and the function ${}_1F_1(a, b; z)$ which appears in the above equation is the confluent hypergeometric function

$${}_1F_1(a, b; z) := \sum_{n \geq 0} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \tag{22}$$

see Chapter 6 in [11] for an extensive collection of results and formulas related to this function.

While the incomplete gamma function is not one of the elementary functions, it can still be easily evaluated everywhere in the complex plane. In fact, numerical routines for evaluating this function are provided in such computational software programs as Maple and Mathematica. One should use different strategies for computing $\gamma(s, z)$ depending on whether $|s|$ and/or $|z|$ is large. However, we will only need to compute $\gamma(s, z)$ for a fixed s (which is real and not large) and for various values of z . In this case the numerical algorithms are based on one of the infinite series expansions (21), if $|z|$ is not large, or on the various asymptotic approximations (and expansions

in continued fractions, such as formula 8.358 in [15]) when $|z|$ is large. We refer to [16] or [33] for all the details.

Finally, we would like to mention that the code for all numerical experiments was written in C++ and the computations were performed on a standard laptop (Intel Core i5 2.6 GHz processor and 4 GB of RAM).

4.2. Numerical example 1: processes with double-sided jumps

For our first numerical experiment, we consider the generalized tempered stable process (see [9]), also known as the KoBoL process, with the Lévy measure truncated at a positive number k :

$$\Pi(dx) = \mathbf{1}_{\{x < 0\}} \hat{C} \hat{\alpha} e^{\hat{\beta} x} |x|^{-1-\hat{\alpha}} dx + \mathbf{1}_{\{0 < x < k\}} C \alpha e^{-\beta x} x^{-1-\alpha} dx. \tag{23}$$

Proposition 2. *Let X be a Lévy process, with the Lévy measure $\Pi(dx)$ given by (23). Then the Laplace exponent of X is given by*

$$\psi(z) = \frac{1}{2} \sigma^2 z^2 + \mu z - \hat{C} \Gamma(1 - \hat{\alpha}) (\hat{\beta} + z)^{\hat{\alpha}} + C \alpha (\beta - z)^\alpha \gamma(-\alpha, k(\beta - z)) + \eta, \tag{24}$$

where η is chosen so that $\psi(0) = 0$.

The proof of Proposition 2 can be found in Section 5. Note that when $z < \beta$ and $k \rightarrow \infty$, then $\gamma(-\alpha, k(\beta - z)) \rightarrow \Gamma(-\alpha)$ (this follows from (20) when $\alpha < 0$, see also formulas (8.356.3) and (8.357.1) in [15]). This confirms the intuitively obvious result that as the cutoff k becomes very large, the Laplace exponent of the truncated process converges to the Laplace exponent of the generalized tempered stable process, see Proposition 4.2 in [9]. Note also that the function $(\beta - z)^\alpha \gamma(-\alpha, k(\beta - z))$ which appears in (24) is an entire function of z , which confirms the fact that $\psi(z)$ is analytic in the half-plane $\text{Re}(z) > 0$. As in the case of the generalized tempered stable processes, when $\sigma > 0$, or $\alpha > 1$, or $\hat{\alpha} > 1$ we have a process of infinite variation. Finally, in the case when $\sigma = 0$ and $\alpha < 1$, $\hat{\alpha} < 1$ we have a process of finite variation, and in this case μ corresponds to the linear drift of this process.

We consider the following values of parameters:

$$\begin{aligned} \sigma &= 1, & \mu &= -2, & C &= \hat{C} = 1, & \alpha &= \hat{\alpha} = 0.5, \\ \beta &= 1, & \hat{\beta} &= 2, & k &= 1. \end{aligned}$$

These parameters give us a process with negative linear drift and jumps of finite variation and infinite activity. Note that since $\sigma = 1$ we have a process with paths of unbounded variation. We also set $q = 1$.

First we compute 1000 roots ζ_n using the method discussed in Section 4.1. Overall, it takes just 0.15 s to compute 1000 roots. The results are presented on Fig. 1. Note that, as expected, the approximation to ζ_n provided by (7) becomes better and better as n increases, although it is not so good for small values of n .

Next, we compute the density $p(x)$ of the supremum $S_{e(q)}$ using the series representation (14). The results are presented in Fig. 2. After we have pre-computed and stored the roots ζ_n , computing 2000 values of $p(x)$ takes just 0.26 s. In order to test the accuracy, we have numerically computed the integral of $p(x)$ on the interval $[0, 10]$ (which should be close to $\mathbb{P}(S_{e(q)} > 0) = 1$). The result is equal to 0.985, if we use 1000 roots and is equal to 0.995, provided that we use 5000 roots.

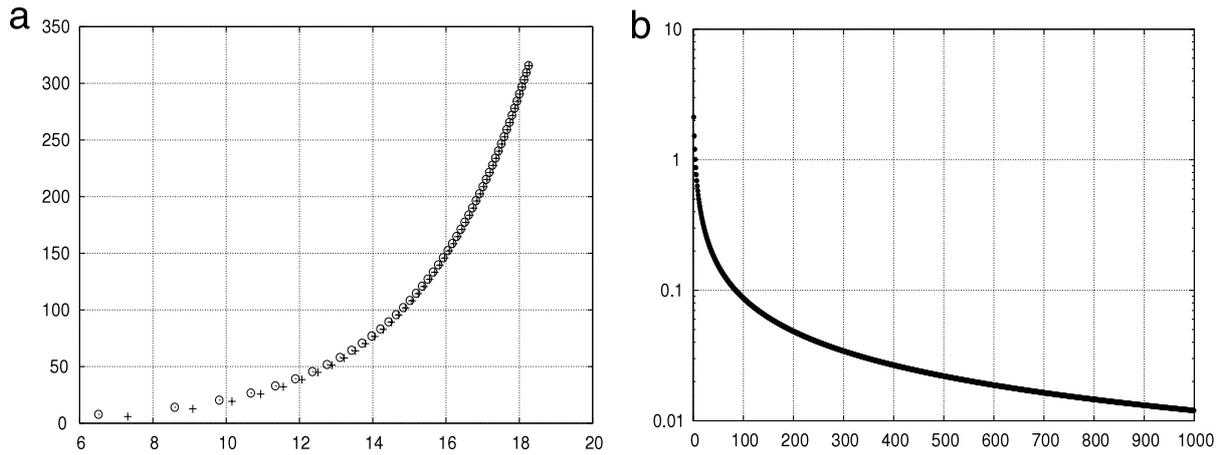


Fig. 1. In graph (a) we present the first fifty complex roots ζ_n (crosses) and their approximations (circles) given by Theorem 2. In graph (b) we present the distance from ζ_n to its approximation (note that we use logarithmic scale for the y-axis).

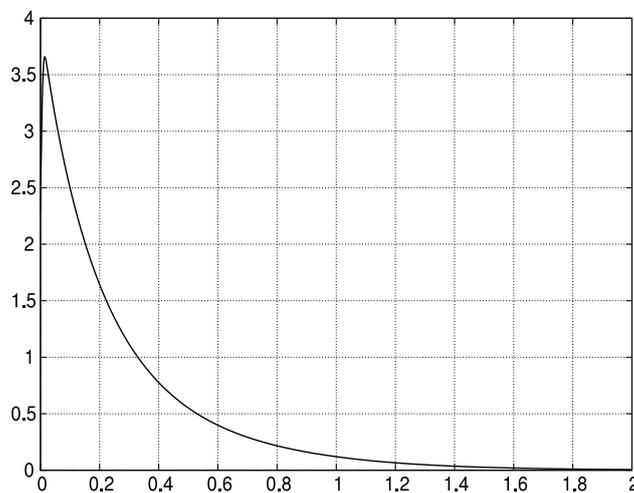


Fig. 2. The density of $S_{e(q)}$ for $q = 1$.

4.3. Numerical example 2: a family of spectrally negative processes

For our second numerical experiment we consider a spectrally negative Lévy process Y , with the Lévy measure defined as follows

$$\Pi_Y(dx) = \mathbf{1}_{\{-k < x < 0\}} C\alpha e^{\beta x} |x|^{-1-\alpha} dx.$$

Note that the dual process $\hat{Y} = -Y$ belongs to the class of processes described in Proposition 2, therefore the Laplace exponent of Y is given by

$$\psi_Y(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + C\alpha(\beta + z)^\alpha \gamma(-\alpha, k(\beta + z)) + \eta, \tag{25}$$

where again η is chosen so that $\psi_Y(0) = 0$.

We fix the following values of parameters

$$\sigma \in \{0, 1\}, \quad \mu = 2, \quad C = 1, \quad \alpha = 0.5, \quad \beta = 1, \quad k = 1, \tag{26}$$

which define a spectrally-negative process with infinite activity/finite variation jumps and paths of finite/infinite variation depending on whether $\sigma = 0$ or $\sigma = 1$. We compute 1000 numbers

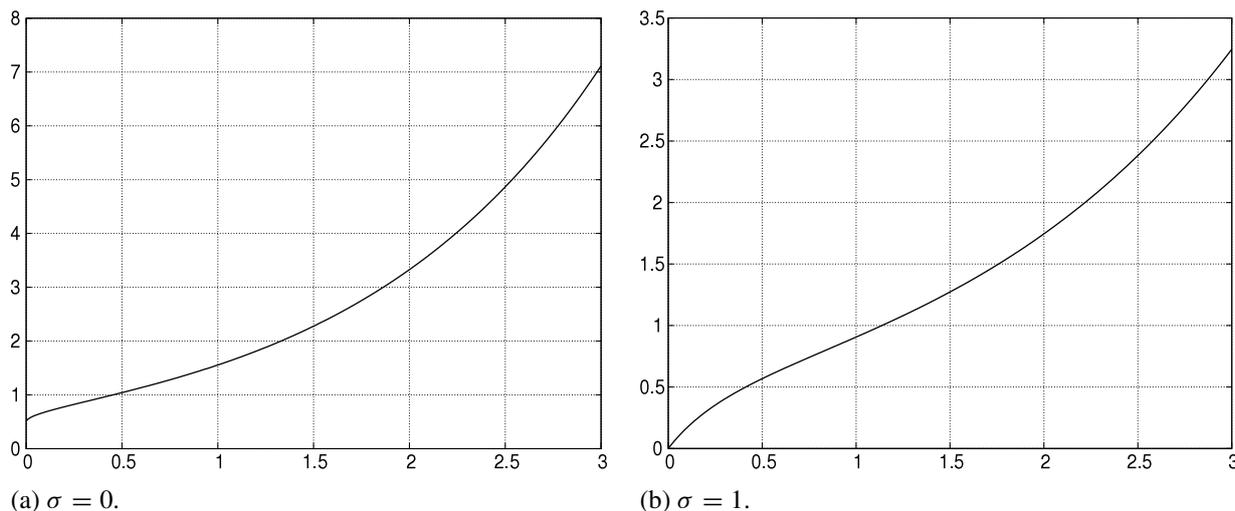


Fig. 3. The scale function $W^{(q)}(x)$ for the parameter set (26).

$\{\zeta_n\}$ using the algorithm presented in Section 4.1. This computation takes 0.06 s. The qualitative behavior of the roots ζ_n is very similar to the one presented on Fig. 1. Computing the scale function via series representation (18) is also very fast: it takes just 0.07 s to compute 1000 values of $W^{(q)}(x)$ for equally spaced points $x \in [0, 3]$.

The results of computation are presented on Fig. 3. From Lemma 8.6 in [25] we know that $W^{(q)}(0) = 0$ if the process Y has infinite variation, and $W^{(q)} = 1/\mu$ if the process has bounded variation (where μ is the linear drift). One can see from Fig. 3 that our numerical results are in perfect agreement with the theoretical prediction. It would also be very interesting to compare the accuracy and performance of this algorithm for evaluating $W^{(q)}(x)$ with various methods used in [23], however we have decided to leave this for future work.

5. Proofs

Lemma 1. *Let $\nu(dx)$ be a finite positive measure such that $\nu((a, +\infty)) = 0$ for some $a \in \mathbb{R}$. Define $k = \inf\{a \in \mathbb{R} : \nu((a, +\infty)) = 0\}$. Then for every $\epsilon > 0$ there exists $\xi = \xi(\epsilon) > 0$ such that for all $z > \xi$ we have*

$$e^{(k-\epsilon)z} < \int_{\mathbb{R}} e^{zx} \nu(dx) < e^{(k+\epsilon)z}. \tag{27}$$

Proof. Assume that $\epsilon > 0$. Since $\nu((k, \infty)) = 0$ we have for all $z > 0$

$$e^{-(k+\epsilon)z} \int_{\mathbb{R}} e^{zx} \nu(dx) = \int_{(-\infty, k]} e^{z(x-k-\epsilon)} \nu(dx) < e^{-\epsilon z} \nu(\mathbb{R}),$$

and the right-hand side in the above inequality goes to zero as $z \rightarrow +\infty$. This proves the upper bound in (27). Similarly,

$$e^{-(k-\epsilon)z} \int_{\mathbb{R}} e^{zx} \nu(dx) > e^{-(k-\epsilon)z} \int_{[k-\epsilon/2, k]} e^{zx} \nu(dx) > e^{\frac{\epsilon}{2}z} \nu([k-\epsilon/2, k]).$$

According to our definition of k , the quantity $\nu([k-\epsilon/2, k])$ is strictly positive, thus the right-hand side in the above inequality goes to $+\infty$ as $z \rightarrow +\infty$, which proves the lower bound in (27). \square

For the proof of **Theorem 1** we need to review some results on entire functions. The interested reader should also consult the excellent books by Levin, [27,28].

A function $f(z) : \mathbb{C} \mapsto \mathbb{C}$ is called an *entire function of exponential type* σ , if $f(z)$ is analytic in \mathbb{C} and

$$\sigma := \limsup_{r \rightarrow +\infty} r^{-1} \ln \left(\max_{|z|=r} |f(z)| \right) \in (0, \infty).$$

Definition 4. By the Cartwright class \mathfrak{C} we mean the class of all entire functions of exponential type satisfying the inequality

$$\int_{\mathbb{R}} \frac{\max(\ln |f(t)|, 0)}{1+t^2} dt < \infty. \tag{28}$$

A good account of the theory of the functions belonging to the Cartwright class can be found in Chapter 5 in [27] and Chapters 16, 17 in [28]. We will need the following fact: if $f \in \mathfrak{C}$, then both the growth of $f(z)$ as $z \rightarrow \infty$ and the distribution of zeros of $f(z)$ are very regular. In particular, **Theorem 2** on page 116 and formula (7) on page 118 in [28] tell us that for $f \in \mathfrak{C}$ the limit

$$\lim_{r \rightarrow +\infty} \frac{\ln |f(re^{i\theta})|}{r} = \begin{cases} \sigma_+ \sin(\theta), & 0 \leq \theta \leq \pi, \\ \sigma_- |\sin(\theta)|, & \pi \leq \theta \leq 2\pi \end{cases} \tag{29}$$

exists for almost all $\theta \in [0, 2\pi]$. Several important results on the distribution of zeros of functions of class \mathfrak{C} are presented in the next theorem. These results can be found in **Theorem 1**, page 127 and **Remark 2**, page 130 in [28]; or in **Theorem 11**, page 251 in [27]. For $0 < \alpha < \pi$ let us denote by $n_+(r, \alpha)$ and $n_-(r, \alpha)$ the numbers of zeros of the function $f(z)$ in the sectors

$$\{z : |z| \leq r, |\arg(z)| \leq \alpha\}, \quad \text{and} \quad \{z : |z| \leq r, |\pi - \arg(z)| \leq \alpha\}, \text{ respectively.}$$

Theorem 4 (Cartwright, Levinson). Let $f \in \mathfrak{C}$ and let $\{a_k\}$, $a_k \neq 0$ be its zero set. Then

- (i) $\sum_k |\operatorname{Im}(1/a_k)| < \infty$;
- (ii) for every $\alpha \in (0, \pi)$ we have $\lim_{r \rightarrow +\infty} r^{-1} n_+(r, \alpha) = \lim_{r \rightarrow +\infty} r^{-1} n_-(r, \alpha) = (\sigma_+ + \sigma_-)/(2\pi)$;
- (iii)

$$f(z) = f(0)e^{-i(\sigma_+ - \sigma_-)z/2} \lim_{R \rightarrow +\infty} \prod_{|a_k| < R} \left(1 - \frac{z}{a_k} \right), \tag{30}$$

where the infinite product converges uniformly on compact subsets of \mathbb{C} .

Proof of Theorem 1. Let us prove (i). The Lévy–Khintchine formula (2) implies that $\psi(z)$ is analytic in the half-plane $\operatorname{Re}(z) > 0$ and convex for $z > 0$. Since $\psi(0) = 0$ and $\psi(z)$ increases exponentially as $z \rightarrow +\infty$ we conclude that there exists a unique simple real solution to $\psi(z) = q$, which we will denote by ζ_0 .

Let us prove that there are no other solutions in the vertical strip $0 \leq \operatorname{Re}(z) < \zeta_0$. From the definition of the Laplace exponent we find

$$e^{t\operatorname{Re}(\psi(z))} = \left| \mathbb{E} \left[e^{zX_t} \right] \right| \leq \mathbb{E} \left[e^{\operatorname{Re}(z)X_t} \right] = e^{t\psi(\operatorname{Re}(z))}.$$

This shows that for $0 \leq \operatorname{Re}(z) < \zeta_0$ we have $\operatorname{Re}(\psi(z) - q) \leq \psi(\operatorname{Re}(z)) - q < 0$, therefore $\psi(z) \neq q$ in this vertical strip and we have proved the first part of [Theorem 1](#).

Next, let us prove (ii), (iii) and (iv). Let us consider the ascending ladder process (L^{-1}, H) and its Laplace exponent $\kappa(q, z)$. Similarly, let $\hat{\kappa}(q, z)$ denote the Laplace exponent of the descending ladder process (\hat{L}^{-1}, \hat{H}) , see Section 6.2 in [\[25\]](#) for the definition and properties of these objects. Let $\Lambda(dt, dx)$ denote the Lévy measure of the bivariate subordinator (L^{-1}, H) (see Section 6.3 in [\[25\]](#)). One can check that $\kappa(q, z)$ can be expressed in the following form

$$\kappa(q, z) = \kappa(q, 0) + az - \int_0^\infty (e^{-zx} - 1) \Lambda^{(q)}(dx) \tag{31}$$

where $a \geq 0$ and

$$\Lambda^{(q)}(dx) := \int_0^\infty e^{-qt} \Lambda(dt, dx).$$

Formula [\(31\)](#) implies that the function $z \mapsto \kappa(q, z)$ is the Laplace exponent of a subordinator with the Lévy measure $\Lambda^{(q)}(dx)$ and drift a , which is killed at the rate $\kappa(q, 0)$. Note that $\Lambda^{(0)}(dx)$ is the Lévy measure of the ascending ladder height process H . The jumps in the process H happen when the process X jumps over the past supremum, thus it is clear that if the jumps of X are bounded from above by k , then the same is true for the process H . Therefore $\Lambda^{(0)}((k, \infty)) = 0$, and since for each Borel set B the quantity $\Lambda^{(q)}(B)$ is decreasing in q we conclude that $\Lambda^{(q)}((k, \infty)) = 0$ for all $q \geq 0$. Therefore we have proved that the support of the measure $\Lambda^{(q)}(dx)$ lies inside the interval $(0, k]$.

Let us define $\tilde{k} := \inf\{x > 0 : \Lambda^{(q)}((x, \infty)) = 0\}$. Note that $\tilde{k} \leq k$, since we have established already that the support of the measure $\Lambda^{(q)}(dx)$ lies inside the interval $(0, k]$. Using [Lemma 1](#), formula [\(31\)](#) and the fact that $\Lambda^{(q)}(dx)$ has finite support we conclude that the function $f(z) := \kappa(q, iz)$ is an entire function of exponential type $\sigma = \tilde{k}$. One can also easily check that for this function $f(z)$ we have $\sigma_+ = \tilde{k}$ and $\sigma_- = 0$, where σ_\pm are defined in [\(29\)](#).

Using the Wiener–Hopf factorization (see [Theorem 6.16](#) in [\[25\]](#)) we find that for $\operatorname{Re}(z) \leq 0$

$$\left| \frac{\kappa(q, 0)}{\kappa(q, -z)} \right| = \left| \mathbb{E} \left[e^{zS_{e(q)}} \right] \right| < \mathbb{E} \left[e^{\operatorname{Re}(z)S_{e(q)}} \right] \leq 1, \tag{32}$$

this shows that the function $\kappa(q, -z)$ has no zeros in the half-plane $\operatorname{Re}(z) \leq 0$. By the same argument applied to $I_{e(q)}$ we conclude that $\hat{\kappa}(q, z)$ has no zeros in the half-plane $\operatorname{Re}(z) \geq 0$. Therefore, using the Wiener–Hopf factorization $q - \psi(z) = \kappa(q, -z)\hat{\kappa}(q, z)$ we find that all zeros of $\kappa(q, -z)$ in the half-plane $\operatorname{Re}(z) > 0$ coincide with the zeros of $q - \psi(z)$. Recall that we have labeled these zeros as $\{\zeta_0, \zeta_n, \bar{\zeta}_n\}_{n \geq 1}$, where $\zeta_n \in \mathcal{Q}_1$ are arranged in the order of increase of the absolute value.

Next, we use [Proposition 2](#) in [\[3\]](#) (page 16) and the fact that $\kappa(q, z)$ is the Laplace exponent of a subordinator to conclude that $\kappa(q, iz) = O(z)$ as $z \rightarrow \infty$, $z \in \mathbb{R}$. Therefore

$$\int_{\mathbb{R}} \frac{\max(\ln(|\kappa(q, iz)|), 0)}{1 + z^2} dz < \infty,$$

and the function $f(z) = \kappa(q, iz)$ belongs to the Cartwright class (see [Definition 4](#)). Using formula [\(30\)](#) and the fact that $\sigma_+ = \tilde{k}$ and $\sigma_- = 0$ we see that $\kappa(q, -z)$ can be factorized as

$$\kappa(q, -z) = \kappa(q, 0)e^{\frac{\tilde{k}}{2}z} \left(1 - \frac{z}{\zeta_0}\right) \prod_{n \geq 1} \left(1 - \frac{z}{\zeta_n}\right) \left(1 - \frac{z}{\bar{\zeta}_n}\right). \tag{33}$$

Moreover, according to [Theorem 4](#), all of the roots ζ_n (except for a set of zero density) lie inside arbitrarily small angle $\pi/2 - \epsilon < \arg(z) < \pi/2$, the density of the roots in this angle exists and is equal to $\tilde{k}/(2\pi)$ and the series $\sum \operatorname{Re}(\zeta_n^{-1})$ converges.

Using [\(31\)](#) and the following result

$$\mathbb{E} \left[e^{-zS_{e(q)}} \right] = \phi_q^+(iz) = \frac{\kappa(q, 0)}{\kappa(q, z)},$$

(see [Theorem 6.16](#) in [\[25\]](#)) we obtain formula [\(5\)](#) for the Wiener–Hopf factors. We see that the proof of [Theorem 1](#) would be complete if we show that $\tilde{k} = k$. Note that we have already established that $\tilde{k} \leq k$, thus we only need to prove that \tilde{k} cannot be strictly smaller than k . This fact seems to be intuitively obvious, as it means that the upper boundary of the support of the Lévy measure of the process X cannot be greater than the upper boundary of the support of the Lévy measure of the ascending ladder height process H . However, we were not able to find a simple probabilistic argument to prove this statement, and we will use an analytic approach instead.

Assume that $\tilde{k} < k$ and define $\epsilon = (k - \tilde{k})/3$. Using the Lévy–Khintchine formula [\(2\)](#) and [Lemma 1](#) one can check that $|\psi(z)| > \exp((k - \epsilon)z)$ for all $z > 0$ large enough. Similarly, using [\(31\)](#), [Lemma 1](#) and the fact that $\Lambda^{(q)}(dx)$ has support on $(0, \tilde{k}]$ we can check that $|\kappa(q, -z)| < \exp((\tilde{k} + \epsilon)z)$ for all $z > 0$ large enough. From the Wiener–Hopf factorization $\hat{\kappa}(q, z) = (q - \psi(z))/\kappa(q, -z)$ we conclude that

$$|\hat{\kappa}(q, z)| > \frac{e^{(k-\epsilon)z}}{e^{(\tilde{k}+\epsilon)z}} = e^{\epsilon z}$$

for all $z > 0$ large enough. But this is not possible, as we know that $\hat{\kappa}(q, z)$ is the Laplace exponent of a subordinator, thus $\hat{\kappa}(q, z) = O(z)$ as $z \rightarrow \infty$, $\operatorname{Re}(z) \geq 0$ (see [Proposition 2](#), page 16 in [\[3\]](#)). Therefore, the inequality $\tilde{k} < k$ is not true, which implies that $\tilde{k} = k$ and ends the proof of parts (ii), (iii) and (iv) of [Theorem 1](#). \square

Recall that $\Delta \arg(f, C)$, which was defined in [\(19\)](#), denotes the change in the argument of $f(z)$ over a curve C . The following result will be used in the proof of [Theorem 2](#).

Proposition 3. *Assume that f and g are analytic on a piecewise smooth curve C and $f(z) \neq 0$ for $z \in C$. If for some $\epsilon \in (0, 1)$ we have $|g(z)| < \epsilon|f(z)|$ for all $z \in C$, then*

$$|\Delta \arg(f + g, C) - \Delta \arg(f, C)| < 4\epsilon. \tag{34}$$

Proof. From the definition of $\Delta \arg(f, C)$ [\(19\)](#) it follows that

$$\Delta \arg(f + g, C) = \Delta \arg(f, C) + \Delta \arg(1 + g/f, C).$$

Due to the condition $|g(z)/f(z)| < \epsilon$ for $z \in C$ we know that the set $\{w = 1 + g(z)/f(z) : z \in C\}$ lies inside the circle of radius ϵ with center at one. Using elementary geometric considerations we check that

$$\max\{|\arg(w)| : |w - 1| < \epsilon\} = \arcsin(\epsilon) < 2\epsilon,$$

thus the change of the argument of any curve lying inside the circle $|w - 1| < \epsilon$ cannot be greater than 4ϵ , which proves [\(34\)](#). \square

Proof of Theorem 2. Let us define $\mathcal{Z} := \{z \in \mathcal{Q}_1 : \psi(z) = q\} = \{\zeta_n\}_{n \geq 1}$ and denote

$$z_n := \frac{1}{k} \left[\ln \left(\left| \frac{B}{A} \right| \right) + (a + b) \ln \left(\frac{2n\pi}{k} \right) \right] + \frac{i}{k} \left[\arg \left(\frac{B}{A} \right) + \left(\frac{1}{2}(a + b) + 2n + 1 \right) \pi \right].$$

First let us prove that every solution to the equation $\psi(z) = q$ which has sufficiently large absolute value must be close to one of z_n . From the asymptotic expansion (6) we find that when $z \rightarrow \infty$, $z \in \mathcal{Z}$ we have

$$e^{kz} = -\frac{B}{A} z^{a+b} (1 + o(1)). \tag{35}$$

Considering the absolute value of both sides of (35) we conclude

$$\operatorname{Re}(z) = \frac{1}{k} \ln \left| \frac{B}{A} \right| + \frac{a + b}{k} \ln |z| + o(1), \quad z \rightarrow \infty, z \in \mathcal{Z}. \tag{36}$$

From the above asymptotic expression it follows that $\operatorname{Re}(z) = O(\ln |z|)$, which in turn implies

$$\arg(z) = \frac{\pi}{2} + o(1), \quad z \rightarrow \infty, z \in \mathcal{Z}. \tag{37}$$

Next, considering the argument of both sides of (35), we find that

$$\arg(e^{kz}) = \arg\left(-\frac{B}{A}\right) + \arg(z^{a+b}) + \arg(1 + o(1)) \pmod{2\pi},$$

and using (37) we conclude that there exists an integer number n such that

$$k \operatorname{Im}(z) = \arg\left(\frac{B}{A}\right) + (2n + 1)\pi + \frac{1}{2}(a + b)\pi + o(1). \tag{38}$$

Finally, asymptotic expressions (36) and (38) imply that

$$\operatorname{Re}(z) = \frac{1}{k} \left[\ln \left(\left| \frac{B}{A} \right| \right) + (a + b) \ln \left(\frac{2n\pi}{k} \right) \right] + o(1),$$

which together with (38) shows that every sufficiently large solution to $\psi(z) = q$ must be close to one of the numbers z_n .

Now let us prove the converse statement: for all n large enough there is always a solution of the equation $\psi(z) = q$ near a point z_n . We set $\epsilon = 1/(16k(a + b))$, assume that n is a large positive integer and consider the following contour $L = L(n) := L_1 \cup L_2 \cup L_3 \cup L_4$, defined as

$$\begin{aligned} L_1 = L_1(n) &:= \left\{ z \in \mathbb{C} : \operatorname{Re}(z) = 0, |\operatorname{Im}(z) - \operatorname{Im}(z_n)| \leq \frac{\pi}{k} \right\}, \\ L_2 = L_2(n) &:= \left\{ z \in \mathbb{C} : \operatorname{Im}(z) = \operatorname{Im}(z_n) - \frac{\pi}{k}, 0 \leq \operatorname{Re}(z) \leq \epsilon n \right\}, \\ L_3 = L_3(n) &:= \left\{ z \in \mathbb{C} : \operatorname{Re}(z) = \epsilon n, |\operatorname{Im}(z) - \operatorname{Im}(z_n)| \leq \frac{\pi}{k} \right\}, \\ L_4 = L_4(n) &:= \left\{ z \in \mathbb{C} : \operatorname{Im}(z) = \operatorname{Im}(z_n) + \frac{\pi}{k}, 0 \leq \operatorname{Re}(z) \leq \epsilon n \right\}. \end{aligned}$$

As we see on Fig. 4(a), L is a rectangle of dimensions $2\pi/k$ and ϵn , which contains exactly one point z_n for n large enough. We assume that this contour is oriented counter-clockwise. Our goal

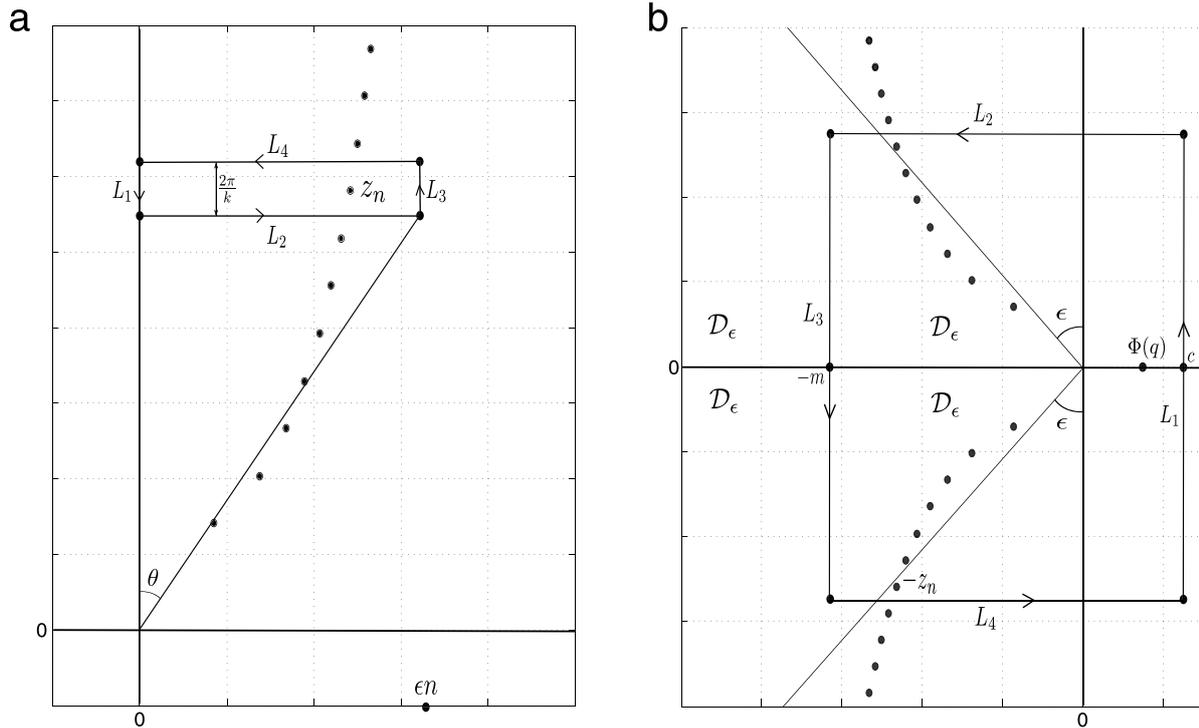


Fig. 4. Illustrations to the proofs of Theorems 2 and 3.

is to prove that $\Delta \arg(\psi(z) - q, L(n)) = 2\pi$ for all n large enough, and our strategy is to show that the change in the argument over L_1, L_2 and L_4 is small, while the change in the argument over L_3 is close to 2π .

First of all, it is clear that the number of zeros of $\psi(z) - q$ inside the contour L is the same as the number of zeros of $F(z) := z^a(\psi(z) - q)$ inside the same contour. Asymptotic expression (6) tells us that

$$F(z) = Ae^{kz} + Bz^{a+b} + o(e^{kz}) + o(z^{a+b}), \quad z \rightarrow \infty, z \in \mathcal{Q}_1. \tag{39}$$

Let us first consider the interval L_1 . Since $\arg(z) = \pi/2$ on this interval, we have $\Delta \arg(z^{a+b}, L_1) = 0$. Eq. (39) implies that $F(z) = Bz^{a+b} + o(z^{a+b})$ when $z \in L_1$ and $n \rightarrow +\infty$, thus we use Proposition 3 and conclude that for all n large enough we have $|\Delta \arg(F, L_1(n))| < 1/4$.

Let us consider the contour L_2 . From the definition of this contour it follows that for all $z \in L_2$

$$\arg\left(\frac{A}{B} \exp\left(kz - \frac{\pi i}{2}(a+b)\right)\right) = 0. \tag{40}$$

Also, looking at Fig. 4(a) one can check that $\pi/2 - \theta \leq \arg(z) \leq \pi/2$ for all $z \in L_2$, where we have defined

$$\theta = \theta(\epsilon, n) = \arctan\left(\frac{\epsilon n}{\text{Im}(z_n) - \pi/k}\right).$$

Note that as $n \rightarrow +\infty$ we have $\theta(\epsilon, n) \rightarrow \arctan(\epsilon k/(2\pi))$, and the latter quantity is smaller than $k\epsilon$. This implies that for all n large enough we have $\theta(\epsilon, n) < k\epsilon$. Thus we have proved that for all n large enough we have $\pi/2 - k\epsilon \leq \arg(z) \leq \pi/2$ when $z \in L_2$, which is equivalent to

$$\left|\arg\left(z^{a+b} \exp\left(-\frac{\pi i}{2}(a+b)\right)\right)\right| < k(a+b)\epsilon = \frac{1}{16}, \quad z \in L_2. \tag{41}$$

Next, we will need the following two facts, which can be easily verified by elementary geometric considerations:

(i) For any real $u > 0$ and any complex number v for which $|\arg(v)| < \pi/2$ we have

$$|\arg(u + v)| \leq |\arg(v)|. \tag{42}$$

(ii) For any two complex numbers u, v we have

$$|u + v| \geq \cos(\arg(u) - \arg(v))(|u| + |v|). \tag{43}$$

From (40) and (41) and property (42) it follows that for all $z \in L_2$ the number

$$w = \frac{A}{B} \exp\left(kz - \frac{\pi i}{2}(a + b)\right) + z^{a+b} \exp\left(-\frac{\pi i}{2}(a + b)\right)$$

lies in the sector $|\arg(w)| < 1/16$. From here we find that

$$|\Delta \arg(Ae^{kz} + Bz^{a+b}, L_2)| < \frac{1}{8}, \tag{44}$$

and at the same time, with the help of property (43) we deduce that for all $z \in L_2$

$$\left|Ae^{kz} + Bz^{a+b}\right| > \cos\left(\frac{1}{16}\right) \left(\left|Ae^{kz}\right| + \left|Bz^{a+b}\right|\right). \tag{45}$$

Next, if $g(z) = o(e^{kz}) + o(z^{a+b})$ as $z \rightarrow \infty$, then

$$g(z) = o\left(\left|Ae^{kz}\right| + \left|Bz^{a+b}\right|\right),$$

and we again can use (34) and (44) with $f(z) := F(z)$ and $g(z)$ defined above, to conclude that for all n large enough

$$|\Delta \arg(F(z), L_2(n)) - \Delta \arg(Ae^{kz} + Bz^{a+b}, L_2)| < 1/8.$$

The above inequality and estimate (44) imply that for all n large enough we have $|\Delta \arg(F(z), L_2(n))| < 1/4$. Using exactly the same technique we obtain an identical estimate the change of argument over $L_4(n)$.

Finally, on the contour L_3 we have $F(z) = A \exp(kz) + o(\exp(kz))$. Since $\Delta \arg(\exp(kz), L_3) = 2\pi$, we use Proposition 3 and conclude that for all n large enough $|\Delta(F, L_3(n)) - 2\pi| < 1/4$. Combining these four estimates we see that for all n large enough we have $|\Delta(F, L(n)) - 2\pi| < 1$, and since we know that $\Delta \arg(F, L)$ must be an integer multiple of 2π we conclude that $\Delta \arg(F, L) = 2\pi$, thus there is exactly one solution to $\psi(z) = q$ inside the contour $L(n)$. From the first part of the proof we know that every sufficiently large solution to $\psi(z) = q$ must be close to z_m for some m , and since by construction there is only one such point z_n inside the contour $L(n)$, we conclude that for every n large enough there is a solution to $\psi(z) = q$ close to z_n . \square

Proof of Proposition 1. Let us assume that $\alpha < 1$ and $\hat{\alpha} < 1$. Then we can take the cutoff function $h(x) \equiv 0$ in (2), and we can rewrite $\psi(z)$ as follows

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + \psi_1(z) + \psi_2(z), \tag{46}$$

where we have denoted

$$\psi_1(z) := \int_{(-\infty, 0)} (e^{zx} - 1) \Pi(dx), \tag{47}$$

$$\psi_2(z) := \int_{(0,k]} (e^{zx} - 1) \Pi(dx). \tag{48}$$

First, let us study the asymptotic behavior of $\psi_1(z)$. If $\hat{\alpha} < 0$ then part (1) of **Definition 2** implies that $\Pi((-\infty, 0)) < \infty$, thus $\psi_1(z) = O(1)$ as $z \rightarrow \infty, \text{Re}(z) \geq 0$. If $0 < \hat{\alpha} < 1$, then part (1) of **Definition 2** implies that for $x \in \mathbb{R}^-$

$$\Pi(dx) = \left(\hat{C}\hat{\alpha}|x|^{-1-\hat{\alpha}} + \sum_{j=1}^{\hat{m}} \hat{C}_j \hat{\alpha}_j |x|^{-1-\hat{\alpha}_j} \right) dx + \nu(dx),$$

where $\nu(dx)$ is a finite measure on \mathbb{R}^- (note that $\nu(dx)$ does not have to be a positive measure). It is clear that

$$\left| \int_{-\infty}^0 (e^{zx} - 1) \nu(dx) \right| < 2|\nu((-\infty, 0))|$$

for $\text{Re}(z) \geq 0$. Using integration by parts we find that for $\text{Re}(z) > 0$

$$\hat{\alpha} \int_{-\infty}^0 (e^{zx} - 1) |x|^{-1-\hat{\alpha}} dx = -\Gamma(1 - \hat{\alpha})z^{\hat{\alpha}}. \tag{49}$$

Combining the above three equations and (47) we conclude that

$$\psi_1(z) = -\hat{C}\Gamma(1 - \hat{\alpha})z^{\hat{\alpha}} + o(z^{\hat{\alpha}}) \tag{50}$$

as $z \rightarrow \infty, \text{Re}(z) > 0$.

Next, let us investigate the asymptotic behavior of $\psi_2(z)$. Let us assume that $n > 0$ (where n is the constant in the **Definition 2**), the proof in the case $n = 0$ is very similar. Since $n > 0$, **Definition 2** implies that the measure $\Pi(dx)$ restricted to \mathbb{R}^+ has a density $\pi(x)$, which belongs to $\mathcal{PC}^n((0, k])$. Let us assume first that $\pi \in C^n((0, k])$, we will relax this assumption later.

First let us consider the case $\alpha < 0$, which is equivalent to $\Pi((0, \infty)) < \infty$. Applying integration by parts n times to (47) we obtain

$$\begin{aligned} \psi_2(z) &= \int_0^k e^{zx} \pi(x) dx - \int_0^k \pi(x) dx \\ &= \sum_{m=0}^{n-1} (-1)^m \left[\pi^{(m)}(k-) e^{kz} z^{-m-1} - \pi^{(m)}(0+) z^{-m-1} \right] \\ &\quad + (-1)^n z^{-n} \int_0^k \pi^{(n)}(x) e^{zx} dx - \int_0^k \pi(x) dx. \end{aligned}$$

Since $\pi^{(n)}(x)$ is continuous we conclude that

$$\int_0^k \pi^{(n)}(x) e^{zx} dx = o(e^{kz})$$

as $z \rightarrow \infty, \text{Re}(z) > 0$. At the same time, due to **Definition 2** we have $\pi^{(m)}(k-) = 0$ for all $m \leq n - 2$. Using the above two results and the fact that $\frac{d}{dx} \Pi^+(x) = -\pi(x)$ we obtain

$$\psi_2(z) = (-1)^n \Pi^{+(n)}(k-) e^{kz} z^{-n} + o(e^{kz} z^{-n}) + O(1), \tag{51}$$

as $z \rightarrow \infty, \operatorname{Re}(z) > 0$. Eq. (51) shows that the exponential term on the right-hand side of (6) comes from the upper boundary of the support of the Lévy measure and from the first non-zero derivative of $\bar{I}^+(x)$ at k^- .

Next, let us assume that $\alpha \in (0, 1)$. Then, according Definition 2, the density of the Lévy measure can be expressed as follows

$$\pi(x) = C\alpha x^{-1-\alpha} + \sum_{j=1}^m C_j \alpha_j x^{-\alpha_j} + g(x),$$

where $g \in C^n([0, k])$. We can rewrite $\psi_2(z)$ as

$$\psi_2(z) = CF(\alpha, k, z) + \sum_{j=1}^m C_j F(\alpha_j, k, z) + \int_0^k e^{zx} g(x) dx - \int_0^k g(x) dx, \tag{52}$$

where we have defined

$$F(\alpha, k, z) := \alpha \int_0^k (e^{zx} - 1) x^{-1-\alpha} dx.$$

Let us obtain an asymptotic expansion of $F(\alpha, k, z)$ as $z \rightarrow \infty, z \in \mathcal{Q}_1$. Expanding $\exp(zx)$ in Taylor series centered at zero and integrating term by term we find that

$$F(\alpha, k, z) = k^{-\alpha} [1 - {}_1F_1(-\alpha, 1 - \alpha; kz)], \tag{53}$$

where ${}_1F_1(a, b; z)$ is the confluent hypergeometric function defined by (22). Applying asymptotic formula (2) on page 278 in [11] we conclude that

$$F(\alpha, k, z) = -\Gamma(1 - \alpha) e^{-\pi i \alpha} z^\alpha + \alpha k^{-1-\alpha} \frac{e^{kz}}{z} \sum_{m=0}^N \frac{(1 + \alpha)_m}{(kz)^m} + O(1) + O(z^{\alpha-1}) + O(e^z z^{-N-2}), \tag{54}$$

as $z \rightarrow \infty, z \in \mathcal{Q}_1$. Formula (54) and our previous result (51) imply that

$$\psi_2(z) = (-1)^n \Pi^{+(n)}(k^-) e^{kz} z^{-n} - \Gamma(1 - \alpha) e^{-\pi i \alpha} z^\alpha + o(e^{kz} z^{-n}) + o(z^\alpha) + O(1), \tag{55}$$

as $z \rightarrow \infty, z \in \mathcal{Q}_1$.

As a final step, let us relax the assumption $\pi \in C^n([0, k])$. Assume that there is a unique point $x_1 \in (0, k)$ at which $\pi^{(n)}(x)$ does not exist (the proof in the general case is exactly the same). According to Definition 2, $\pi \in C^{n-2}(\mathbb{R}^+)$, thus $\pi^{(m)}(x_1^-) = \pi^{(m)}(x_1^+)$ for $m \leq n - 2$. Applying integration by parts n times on each subinterval $(0, x_1)$ and (x_1, k) we would obtain an expression (55) plus an extra term of the form

$$h(z) = (-1)^{n-1} \left[\pi^{(n-1)}(x_1^-) - \pi^{(n-1)}(x_1^+) \right] e^{x_1 z} z^{-n}.$$

However, it is easy to see that $h(z) = o(e^{kz} z^{-n}) + o(z^\alpha)$ as $z \rightarrow \infty, z \in \mathcal{Q}_1$. This is true since in the domain $\mathcal{D} = \{\operatorname{Re}(z) < \ln(\ln(\operatorname{Im}(z))), \operatorname{Im}(z) > e\}$ we have $|\exp(x_1 z)| = \exp(x_1 \operatorname{Re}(z)) = O(\ln |z|) = o(z^\alpha)$, while in the domain $\mathcal{Q}_1 \setminus \mathcal{D}$ we have $\operatorname{Re}(z) \rightarrow \infty$ when $z \rightarrow \infty$, which implies $\exp(x_1 z) z^{-n} = o(\exp(kz) z^{-n})$.

Formula (46) and asymptotic expressions (55), (50) imply that $\psi(z)$ satisfies (6) with coefficients A, a, B and b as in Proposition 1, except that there would be an extra term $O(1)$ in the

right-hand side of (6), which comes from (55). According to our assumption, the process X is not a compound Poisson process, thus the constant b defined in Proposition 1 is strictly positive, and $O(1) = o(z^b)$, therefore this extra term can be absorbed into $o(z^b)$. This ends the proof in the case $\alpha < 1$ and $\hat{\alpha} < 1$.

In the case when one or both of $\alpha, \hat{\alpha}$ are greater than one the proof is identical, except that we will have to do one extra integration by parts for proving (50). The details are left to the reader. \square

Proof of Theorem 3. Let us denote

$$z_n := \frac{1}{k} \left[\ln \left(\left| \frac{B}{A} \right| \right) + (a + b) \ln \left(\frac{2n\pi}{k} \right) \right] + \frac{i}{k} \left[\arg \left(\frac{B}{A} \right) + \left(\frac{1}{2}(a + b) + 2n + 1 \right) \pi \right].$$

Due to the Definition 2, the Lévy measure $\Pi(dx)$ can only have a finite number of atoms. From Corollary 2.5 in [23] we find that $W^{(q)}(x)$ can only have a finite number of points where it is not differentiable. Thus we can use (15) and the Bromwich integral formula to conclude that for any $c > \Phi(q)$

$$W^{(q)}(x) = \frac{1}{2\pi i} \int_{c+i\mathbb{R}} \frac{e^{zx}}{\psi_Y(z) - q} dz. \tag{56}$$

For $n > 0$ and $m < 0$ we define the contour $L = L(n, m) := L_1 \cup L_2 \cup L_3 \cup L_4$, where

$$L_1 = L_1(n) := \{\text{Re}(z) = c, -\text{Im}(z_n) - \pi/k < \text{Im}(z) < \text{Im}(z_n) + \pi/k\},$$

$$L_2 = L_2(n, m) := \{\text{Im}(z) = \text{Im}(z_n) + \pi/k, m < \text{Re}(z) < c\},$$

$$L_3 = L_3(n, m) := \{\text{Re}(z) = m, -\text{Im}(z_n) - \pi/k < \text{Im}(z) < \text{Im}(z_n) + \pi/k\},$$

$$L_4 = L_4(n, m) := \{\text{Im}(z) = -\text{Im}(z_n) - \pi/k, m < \text{Re}(z) < c\}.$$

This contour is shown on Fig. 4(b). We assume that L is oriented counter-clockwise. Using the residue theorem we deduce

$$\int_L \frac{e^{zx}}{\psi_Y(z) - q} dz = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{e^{-\zeta_0 x}}{\psi'(-\zeta_0)} + 2 \sum \text{Re} \left[\frac{e^{-\zeta_j x}}{\psi'(-\zeta_j)} \right], \tag{57}$$

where the summation is over all $j \geq 1$, such that $-\zeta_j$ lie inside the contour L .

First, assume that n is fixed and let us consider what happens as $m \rightarrow -\infty$. According to the asymptotic relation (16), as $\text{Re}(z) \rightarrow -\infty$ the function $\psi_Y(z)$ increases exponentially (uniformly in every horizontal strip $|\text{Im}(z)| < C$). In particular, for m large enough we would have $|\psi_Y(z) - q| > 1$ for all $z \in L_3(n, m)$, which implies

$$\left| \int_{L_3(n,m)} \frac{e^{zx}}{\psi_Y(z) - q} dz \right| \leq \int_{L_3(n,m)} \left| \frac{e^{zx}}{\psi_Y(z) - q} \right| \times |dz| < \int_{L_3(n,m)} e^{xm} \times |dz| = (2\text{Im}(z_n) + 2\pi/k) e^{mx},$$

and for every $x > 0$ the right-hand side converges to zero as $m \rightarrow -\infty$.

Our next goal is to let $n \rightarrow +\infty$ and to prove that the integrals over the two horizontal half-lines $L_2(n, -\infty)$ and $L_4(n, -\infty)$ in (57) disappear. In order to achieve this we will need to obtain good upper bounds on $\psi(z)$ on these horizontal half-lines. Let us consider first the contour

$L_2(n, -\infty)$. We will prove that there exists a constant C such that $|\psi_Y(z)| > C|\text{Im}(z_n)|$ for all $z \in L_2(n, -\infty)$.

Assume that $\epsilon > 0$ is a small number and define a domain

$$\mathcal{D}_\epsilon = \{z \in \mathbb{C} : |\arg(z)| > \pi/2 + \epsilon\},$$

see Fig. 4(b). Let $L_5 = L_5(n) = L_2(n, -\infty) \cap \mathcal{D}_\epsilon$ and $L_6 = L_6(n) = L_2(n, -\infty) \setminus \mathcal{D}_\epsilon$. Following the same steps as in the proof of Theorem 2 (see estimate (45)) we find that there exists a constant C_1 (which does not depend on n or ϵ) such that for all n large enough we have for all $z \in L_5(n)$

$$\begin{aligned} |\psi_Y(z)| &> \cos(C_1\epsilon) \left(|Ae^{kz}z^{-a}| + |Bz^b| \right) > \cos(C_1\epsilon)|B||z|^b \\ &> \cos(C_1\epsilon)|B|\text{Im}(z_n)^b > \cos(C_1\epsilon)|B|\text{Im}(z_n) \end{aligned} \tag{58}$$

where in the last estimate we have used the fact that $b \geq 1$ (see (17)).

Next, it can be easily seen from the Fig. 4(b) that for all z in the domain \mathcal{D}_ϵ we have

$$\text{Re}(z) < -|z| \sin(\epsilon), \tag{59}$$

therefore $|z|^b = o(\exp(-kz)z^{-a})$ when $z \rightarrow \infty$, $z \in \mathcal{D}_\epsilon$. This fact and the asymptotic formula (16) show that there exists a constant C_2 such that for all $z \in \mathcal{D}_\epsilon$ large enough we have $|\psi_Y(z)| > C_2|\exp(-kz)z^{-a}|$. Therefore, for all n large enough we have

$$|\psi_Y(z)| > C_2 e^{k|\text{Re}(z)|} |z|^{-a}, \quad z \in L_6(n).$$

Using the above estimate and (59) we find

$$|\psi_Y(z)| > C_2 \sin(\epsilon)^a e^{k|\text{Re}(z)|} |\text{Re}(z)|^{-a}, \quad z \in L_6(n). \tag{60}$$

Next, as n increases to $+\infty$, the real part of any $z \in L_6(n)$ decreases to $-\infty$ (see Fig. 4(b)), thus for all n large enough we have $\exp(k|\text{Re}(z)|/2) > |\text{Re}(z)|^a$ for all $z \in L_6(n)$. At the same time, from the Fig. 4(b) we see that for all $z \in L_6(n)$ it is true that $|\text{Re}(z)| > \tan(\epsilon)|\text{Im}(z)| = \tan(\epsilon)(\text{Im}(z_n) + \pi/k)$. Using this fact and (60) we find that there exists a constant $C_3 = C_3(\epsilon)$ such that for all n large enough

$$\begin{aligned} |\psi_Y(z)| &> C_2 \sin(\epsilon)^a e^{\frac{k}{2}|\text{Re}(z)|} |\text{Re}(z)|^{-a} e^{\frac{k}{2}|\text{Re}(z)|} \\ &> C_2 \sin(\epsilon)^a e^{\frac{k}{2} \tan(\epsilon)\text{Im}(z_n)} > C_3 \text{Im}(z_n), \quad z \in L_6(n). \end{aligned} \tag{61}$$

Combining (58) and (61) we conclude that there exists a constant $C > 0$, such that for all n large enough we have

$$|\psi_Y(z)| > C\text{Im}(z_n), \quad z \in L_2(n, -\infty).$$

A similar estimate for $L_4(n, -\infty)$ can be obtained in the same way.

Thus setting $z = z(u) := u + i(\text{Im}(z_n) + \pi/k)$ we obtain

$$\begin{aligned} \left| \int_{L_2(n, -\infty)} \frac{e^{zx}}{\psi_Y(z) - q} dz \right| &= \left| \int_{-\infty}^c \frac{e^{z(u)x}}{\psi_Y(z(u)) - q} du \right| < \int_{-\infty}^c \frac{|e^{z(u)x}|}{|\psi_Y(z(u)) - q|} du \\ &< \int_{-\infty}^c \frac{e^{ux}}{C|\text{Im}(z_n)| - q} du = \frac{x^{-1}e^{cx}}{C|\text{Im}(z_n)| - q} \end{aligned}$$

and the right hand side converges to zero as $n \rightarrow +\infty$. Similarly, the integral over L_4 vanishes. Thus as $n \rightarrow +\infty$ formula (57) becomes

$$\int_{c+i\mathbb{R}} \frac{e^{zx}}{\psi_Y(z) - q} dz = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{e^{-\zeta_0 x}}{\psi'(-\zeta_0)} + 2 \sum_{n \geq 1} \operatorname{Re} \left[\frac{e^{-\zeta_n x}}{\psi'(-\zeta_n)} \right]$$

and the left-hand side is equal to $W^{(q)}(x)$ due to Bromwich integral formula (56).

Next, from Proposition 1 we know that the asymptotic formula for $\psi'_Y(z)$ can be obtained by differentiating (16). Therefore, using the asymptotic expression (7) for ζ_n we find that

$$|\psi'(-\zeta_n)| = k|B| \left(\frac{2n\pi}{k} \right)^b + o(n^b).$$

Similarly, from (7) we find that there exists a constant c such that

$$|e^{-\zeta_n x}| \sim cn^{-\frac{x}{k}(a+b)}, \quad n \rightarrow +\infty,$$

thus the terms of the series in the right-hand side of (18) decrease as $n^{-b-x(a+b)/k}$. According to (17) we always have $b \geq 1$, which implies that the series in the right-hand side of (18) converges on \mathbb{R}^+ and uniformly on $[\epsilon, \infty)$ for each $\epsilon > 0$. \square

Proof of Proposition 2. First, let us assume that $\alpha < 1$ and $\hat{\alpha} < 1$. We start with the Lévy–Khintchine formula (2) with the cutoff function $h(x) \equiv 0$, which gives us

$$\begin{aligned} \psi(z) &= \frac{1}{2}\sigma^2 z^2 + \mu z + \hat{C}\hat{\alpha} \int_{-\infty}^0 (e^{zx} - 1) e^{\hat{\beta}x} |x|^{-1-\hat{\alpha}} dx \\ &\quad + C\alpha \int_0^k (e^{zx} - 1) e^{-\beta x} x^{-1-\alpha} dx. \end{aligned} \tag{62}$$

The first integral in (62) can be evaluated as follows:

$$\begin{aligned} \hat{C}\hat{\alpha} \int_{-\infty}^0 (e^{zx} - 1) e^{\hat{\beta}x} |x|^{-1-\hat{\alpha}} dx &= \hat{C}\hat{\alpha} \int_{-\infty}^0 (e^{(z+\hat{\beta})x} - 1) |x|^{-1-\hat{\alpha}} dx \\ &\quad - \hat{C}\hat{\alpha} \int_{-\infty}^0 (e^{\hat{\beta}x} - 1) |x|^{-1-\hat{\alpha}} dx \\ &= -\hat{C}\Gamma(1 - \hat{\alpha})(\hat{\beta} + z)^{\hat{\alpha}} + \hat{C}\Gamma(1 - \hat{\alpha})\hat{\beta}^{\hat{\alpha}}, \end{aligned}$$

where we have used (49) in the final step. Similarly, the second integral in (62) can be evaluated with the help of (21) and (53):

$$\begin{aligned} C\alpha \int_0^k (e^{zx} - 1) e^{-\beta x} x^{-1-\alpha} dx &= C\alpha \int_0^k (e^{(z-\beta)x} - 1) x^{-1-\alpha} dx \\ &\quad - C\alpha \int_0^k (e^{-\beta x} - 1) x^{-1-\alpha} dx \\ &= CF(\alpha, k, z - \beta) - CF(\alpha, k, -\beta) \\ &= -Ck^{-\alpha} {}_1F_1(-\alpha, 1 - \alpha, -k(\beta - z)) \\ &\quad + Ck^{-\alpha} {}_1F_1(-\alpha, 1 - \alpha, -k\beta) \\ &= C\alpha(\beta - z)^\alpha \gamma(-\alpha, k(\beta - z)) - C\alpha\beta^\alpha \gamma(-\alpha, k\beta). \end{aligned}$$

This ends the proof in the case $\alpha < 1$ and $\hat{\alpha} < 1$. When $\alpha > 1$ or $\hat{\alpha} > 1$ the proof would be very similar, the only difference is that we would use the cutoff function $h(x) \equiv 1$ in (2) and perform an extra integration by parts in (49). We leave all the details to the reader. \square

Acknowledgments

The first author's research supported by the Natural Sciences and Engineering Research Council of Canada.

The second author's research supported by the University Grants Committee of HKSAR of China and the Department of Mathematics of HKUST.

References

- [1] S. Asmussen, *Ruin Probabilities*, World Scientific, Singapore, 2000.
- [2] S. Asmussen, F. Avram, M.R. Pistorius, Russian and American put options under exponential phase-type Lévy models, *Stoch. Proc. Appl.* 109 (2004) 79–111.
- [3] J. Bertoin, *Lévy Processes*, Cambridge University Press, 1996.
- [4] M.E. Caballero, L. Chaumont, Conditioned stable Lévy processes and the Lamperti representation, *J. Appl. Probab.* 43 (2006) 967–983.
- [5] M.E. Caballero, J.C. Pardo, J.L. Perez, On the Lamperti stable processes, *Probab. Math. Statist.* 30 (1) (2010) 1–28.
- [6] M.E. Caballero, J.C. Pardo, J.L. Perez, Explicit identities for Lévy processes associated to symmetric stable processes, *Bernoulli* 17 (1) (2011) 34–59.
- [7] N. Cai, On first passage times of a hyper-exponential jump diffusion process, *Operations Research Letters* 37 (2) (2009) 127–134.
- [8] L. Chaumont, A.E. Kyprianou, J.C. Pardo, Some explicit identities associated with positive self-similar Markov processes, *Stoch. Proc. Appl.* 119 (3) (2009) 980–1000.
- [9] R. Cont, P. Tankov, *Financial Modeling with Jump Processes*, Chapman & Hall, 2004.
- [10] M. Dellnitz, O. Schutze, Q. Zheng, Locating all the zeros of an analytic function in one complex variable, *J. Comput. Appl. Math.* 138 (2) (2002) 325–333.
- [11] *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, 1953.
- [12] S. Fourati, Explicit solutions of the exit problem for a class of Lévy processes. Applications to the pricing of double barrier options, *Stoch. Proc. Appl.* 122 (3) (2012) 1034–1067.
- [13] F. Hubalek, A.E. Kyprianou, Old and new examples of scale functions for spectrally negative Lévy processes, in: R. Dalang, M. Dozzi, F. Russo. (Eds.), *Sixth Seminar on Stochastic Analysis, Random Fields and Applications*, in: *Progress in Probability*, Birkhäuser, 2010, pp. 119–146.
- [14] M. Jeannin, M.R. Pistorius, A transform approach to calculate prices and greeks of barrier options driven by a class of Lévy processes, *Quant. Finance* 10 (6) (2010) 629–644.
- [15] A. Jeffrey (Ed.), *Table of Integrals, Series and Products*, seven ed., Academic Press, 2007.
- [16] W.B. Jones, W.J. Thron, On the computation of incomplete gamma functions in the complex domain, *J. Comput. Appl. Math.* 12–13 (1985) 401–417.
- [17] S. Kou, A jump diffusion model for option pricing, *Manag. Sci.* (2002) 1086–1101.
- [18] A. Kuznetsov, Wiener–Hopf factorization and distribution of extrema for a family of Lévy processes, *Ann. Appl. Probab.* 20 (5) (2010) 1801–1830.
- [19] A. Kuznetsov, Wiener-Hopf factorization for a family of Lévy processes related to theta functions, *J. Appl. Probab.* 47 (4) (2010) 1023–1033.
- [20] A. Kuznetsov, Analytic proof of Pecherskii–Rogozin identity and Wiener–Hopf factorization, *Theory Probab. Appl.* 55 (3) (2011) 432–443.
- [21] A. Kuznetsov, A.E. Kyprianou, J.C. Pardo, Meromorphic Lévy processes and their fluctuation identities, *Ann. Appl. Probab.* (2010) (in press).
- [22] A. Kuznetsov, A.E. Kyprianou, J.C. Pardo, K. van Schaik, A Wiener–Hopf Monte Carlo simulation technique for Lévy processes, *Ann. Appl. Probab.* 21 (6) (2011) 2171–2190.
- [23] A. Kuznetsov, A.E. Kyprianou, V.M. Rivero, The theory of scale functions for spectrally negative Lévy processes, in: *Levy Matters*, in: *Lecture Notes in Mathematics*, Springer, 2011 (in press).

- [24] A. Kuznetsov, M. Morales, Computing the finite-time expected discounted penalty function for a family of Lévy risk processes, *Scand. Actuar. J.*, in press (<http://dx.doi.org/10.1080/03461238.2011.627747>). Available online at <http://www.tandfonline.com/doi/abs/10.1080/03461238.2011.627747>.
- [25] A.E. Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer, 2006.
- [26] A.E. Kyprianou, J.C. Pardo, V.M. Rivero, Exact and asymptotic n -tuple laws at first and last passage, *Ann. Appl. Probab.* 20 (2) (2010) 522–564.
- [27] B.Ya. Levin, *Distribution of Zeros of Entire Functions*, in: *Translations of Mathematical Monographs*, No. 5, Amer. Math. Soc., 1980.
- [28] B.Ya. Levin, *Lectures on Entire Functions*, in: *Translations of Mathematical Monographs*, No. 150, Amer. Math. Soc., 1996.
- [29] A.L. Lewis, E. Mordecki, Wiener–Hopf factorization for Lévy processes having positive jumps with rational transforms, *J. Appl. Probab.* 45 (1) (2008) 118–134.
- [30] P. Patie, Exponential functional of a new family of Lévy processes and self-similar continuous state branching processes with immigration, *Bull. Sci. Math.* 133 (4) (2009) 355–382.
- [31] E. Pecherskii, B.A. Rogozin, On joint distribution of random variables associated with fluctuations of a process with independent increments, *Theory Probab. Appl.* 14 (1969) 410–423.
- [32] M. Pistorius, On maxima and ladder processes for a dense class of Lévy process, *J. Appl. Probab.* 43 (1) (2006) 208–220.
- [33] S. Winitzki, Computing the Incomplete Gamma Function to Arbitrary Precision, in: *Lecture Notes in Computer Science*, vol. 2668, Springer Verlag, 2003, pp. 790–799.
- [34] J.C. Yakoubsohn, Numerical analysis of a bisection-exclusion method to find zeros of univariate analytic functions, *J. Complexity* 21 (5) (2005) 652–690.
- [35] X. Ying, I.N. Katz, A reliable argument principle algorithm to find the number of zeros of an analytic function in a bounded domain, *Numer. Math.* 53 (1988) 143–163.